JOINT PROGRAM EXAM, MAY 2011

REAL ANALYSIS

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this exam m and m_2 denote one-dimensional and two-dimensional Lebesgue measures, respectively. 'Measurable' is short for 'Lebesgue-measurable'. Instead of dm we sometimes write dx or dy, referring to the variable to be integrated. $L_p(a, b)$ denotes the L_p space with respect to m on the interval (a, b), $-\infty \leq a < b \leq \infty$.

Exam consists of 7 problems. Do ALL of them.

(1) Are the following statements true or false? Justify! If a statement is false, provide a counter-example.

a) If A is an open subset of $[0,1]$ then $m(\bar{A}) = m(A)$. Here $\bar{A}$ is the closure of the set.

b) If A is a subset of $[0,1]$ such that $m(int(A)) = m(\bar{A})$ then A is measurable. Here $int(A)$ is the interior of the set and $\bar{A}$ is the closure of the set.

(2) Let

$$f_n(x) = \frac{1}{1 + x^{\frac{\sqrt{n}}{\ln(n+2011)}}}, \quad x \geq 0.$$

Find $\lim\limits_{n\to\infty} \int\limits_0^\infty f_n(x)\,dx$ and provide full justification.

(3) Denote $J = [0,1] \times [0,1]$. Let h be a real-valued function on J given by

$h(x,y) = 1$ if xy is rational and $h(x,y) = x - y$ otherwise.

Find $\int\limits_J h(x,y)\,dm_2$ and provide full justification.

(4) Give an example for each of the following objects or an explanation why it does not exist.

a) Function f such that $f \in L_1(-\infty,\infty) \cap L_3(-\infty,\infty)$, but $f \notin L_2(-\infty,\infty)$.

b) Function f such that $f \in L_1(-\infty,\infty)$, $f \in L_3(a,b)$ for any *finite* interval (a,b), but $f \notin L_2(-\infty,\infty)$.

(5) Prove that a measurable function $f(x)$ belongs to $L_1(0,1)$ if and only if

$$\sum_{n=1}^{\infty} 2^n \cdot m\{x \in [0,1] : \ |f(x)| \geq 2^n\} < \infty.$$

(6) Let f be a real-valued function on $[0,1]$. Assume that $f(x)$ is differentiable at every point $x \in [0,1]$ (assuming one-side derivatives at boundary points of the interval). Assume also that $|f'(x)| \leq C < \infty$ for all $x \in [0,1]$. Prove that f is absolutely continuous on $[0,1]$.

(7) Let $f \in L_1(-\infty, \infty)$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of strictly positive numbers, i.e. $a_n > 0$ for any n, such that $\sum\limits_{n=1}^{\infty} a_n = 1$. Prove that there exists a partition of $\mathbb{R}$ into measurable sets $\{E_n\}_{n=1}^{\infty}$ such that $\int_{E_n} f(x)\,dm = a_n \int_{-\infty}^{\infty} f(x)\,dm$ for all n.