University of Alabama System
Doctoral Program in Applied Mathematics
Joint Program Exam in Real Analysis

May 15, 2007

Instructions

You may take up to three and a half hours to complete the exam. *Work seven out of the eight problems.* Completeness in your answers is very important. Justify your steps by referring to theorems by name, when appropriate, or by providing a brief theorem statement. You do not need to reprove the theorems you use. An essentially complete and correct solution to one problem will gain more credit, than solutions to two problems, each of which is "half correct". Write the last four digits of your student ID number and problem number on every page.

Notation

Throughout the exam, $\mathbb{R}$ stands for the set of real numbers. Notation such as $\int_{[0,1]} f$, $\int_{[0,1]} f(x)dx$, etc. is used for the Lebesgue integral.

1. Let $\mathcal{A}$ denote the set of real numbers in $[0,1]$, whose decimal expansion does not contain any $1's$. So $x \in \mathcal{A}$ means that $x = .a_1 a_2 \ldots$, where $a_i \neq 1$, for any $i = 1, 2, \ldots$. Find the Lebesgue measure of $\mathcal{A}$.

2. Let g denote a Lebesgue measurable function on $[0,1]$ with $0 \leq g(x) \leq 3$ for $x \in [0,1]$. Show that if $\int_{[0,1]} g(x)\,dx = 3$, then $g(x) = 3$ a.e. on $[0,1]$.

3. Find
$$\lim_{n\to\infty} \int_{[0,n]} \left(\frac{\sin(x)}{x}\right)^n dx.$$
You can use the fact that $|\sin(x)| < x$ for all $x > 0$.

4. Prove that the function $y = \sqrt[3]{x}$ is absolutely continuous on the interval $[-1,1]$.

5. Prove that
$$\int_0^\infty e^{-3x/2}\sqrt{x}\,dx \leq \frac{\sqrt{2}}{2}$$

6. Let E be a measurable subset of $\mathbb{R}^n$, and let A be a non-measurable subset of $\mathbb{R}^k$. Prove that $E \times A$ is measurable if and only if $m_n(E) = 0$. (Here m_n is the Lebesgue measure in $\mathbb{R}^n$.)

7. Let f be a fixed non-negative Lebesgue integrable function on $\mathbb{R}^2$. For any Lebesgue measurable set $E \subseteq \mathbb{R}^2$, define $\mu(E) = \int_E f$ (integral with respect to the Lebesgue measure on $\mathbb{R}^2$).
   (a) Prove that μ is a measure on the σ-algebra $\mathcal{M}$ of all Lebesgue measurable subsets of $\mathbb{R}^2$.
   (b) Give an example of a measure on $\mathcal{M}$ that cannot be obtained by the construction in part (a). Justify your assertion.

8. Let f be Lebesgue integrable on $\mathbb{R}$. Assume that $\int_{\mathbb{R}} f\,dx \neq 0$. Prove that there exists $a \in \mathbb{R}$ such that
$$\int_{(-\infty,a]} f = \int_{[a,\infty)} f$$