JOINT PROGRAM EXAM

REAL ANALYSIS

MAY, 1999

Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name, when appropriate, or by providing a brief theorem statement. An essentially complete and correct solution to one problem will gain more credit, than solutions to two problems each of which is "half correct".

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Notation: Throughout the exam the symbol $m(E)$ refers to Lebesgue measure of the set E and $\mathbf{R}$ stands for the real numbers. The notation $\int_{[0,1]} f(x)dx$ is used for the Lebesgue integral of $f(x)$, while the Riemann integral is denoted by $\int_0^1 f(x)dx$.

1. Prove that if $m^*(E) = 0$, $E \subset \mathbf{R}^n$, then E is Lebesgue measurable ($m^*(E)$ is the outer measure of E).

2. Let $f : [0,1] \to [0,1]$ and $f \in C^1[0,1]$. Use the definitions of Lebesgue measure and Riemann integral to show that the Lebesgue measure of the domain $A = \{(x,y),\ 0 \leq x \leq 1,\ 0 \leq y \leq f(x)\}$ in $\mathbf{R}^2$ is given by the Riemann integral of $f(x)$, i.e.,

$$m(A) = \int_0^1 f(x)dx. \tag{1}$$

3. Let f be a fixed non-negative Lebesgue integrable function on $\mathbf{R}^n$. For any Lebesgue measurable set $E \subseteq \mathbf{R}^n$, define $\mu(E) = \int_E f dx$ (integral with respect to the Lebesgue measure on $\mathbf{R}^n$).

(i) Prove that μ is a measure on the σ-algebra $\mathcal{M}$ of all Lebesgue measurable subsets on $\mathbf{R}^n$.

(ii) Give an example of a measure on $\mathcal{M}$, which can not be obtained by the construction given above. Justify.

4.

(i) Let $f : \mathbf{R}^k \to \mathbf{R}^m$ be Lebesgue measurable, and $g : \mathbf{R}^m \to \mathbf{R}^l$ be continuous. Prove that $g \circ f : \mathbf{R}^k \to \mathbf{R}^l$ is measurable.

(ii) Prove that any monotonic function on (a,b) is Lebesgue measurable.

5. Let $f \in C^1[0,1]$ and positive. Prove that the Riemann and Lebesgue integrals coincide, i.e.,

$$\int_0^1 f(x)dx = \int_{[0,1]} f(x)dx. \tag{2}$$

6. Find the limit

$$\lim_{n\to 0} \int_{[0,1]} \cos(x^n)dx.$$

Be sure you justify all steps.

7. Let $f \in L^1(\mathbf{R}, m)$ and $E_1 \subseteq E_2 \subseteq ...$ be measurable subsets of R. Prove that

$$\lim_{k\to\infty} \int_{E_k} f dx$$

exists, $\int_{E_k} f dx$ being the Lebesgue integrals.

8. Each of the problems below describes a mathematical object with certain properties. If the object exists, give an example. If it does not, give a theorem and/or a short explanation that proves that it does not exist:

(i) An absolutely continuous function f defined on [0,1] and a sequence of subsets E_n of $[0,1]$ such that

$$\frac{m(f(E_n))}{m(E_n)} > n.$$

(ii) A sequence of measurable functions $f_n : [0,1] \to [0,\infty]$ such that

$$\int_{[0,1]} \liminf f_n dx > \liminf \int_{[0,1]} f_n dx.$$

(iii) A sequence of measurable functions $f_n : [0,1] \to [0,\infty]$ such that

$$\int_{[0,1]} \liminf f_n dx < \liminf \int_{[0,1]} f_n dx.$$

9.

(i) Prove that any function of bounded variation can be represented as a difference of two nondecreasing functions.

(ii) Prove or disprove: Any function of bounded variation can be represented as a difference of two *strictly* increasing functions.

10.

(i) Prove that

$$\int_0^{\pi/2} \sqrt{xsinx}dx \leq \frac{\pi}{2\sqrt{2}} \tag{3}$$

(Hint: Hölder's inequality).

(ii)Prove that in fact we have the *strict* inequality in (3).