Joint Program Exam in Real Analysis

September 2009

Instructions:

1. You may take up to three and a half hours to complete the exam.
2. You need to work out all 8 problems in the test for full credit.
3. Completeness in your answers is very important. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is “half correct”.
4. Justify all your claims. When appropriate, refer to a theorem by name or by providing a complete statement of its content.

Notation:

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R} := \mathbb{R}^1$. $\{f_k\}$ denotes a sequence of functions for $k = 1, 2, \cdots$. All integrals occurring are Lebesgue integrals.

1. Are the following statements true or false? Justify!

(i) If $f \in L^1[0,\infty)$ and $f \geq 0$, then $\lim_{x\to\infty} f(x) = 0$.

(ii) If $\lim_{x\to\infty} f(x) = 0$ and $f \geq 0$ on $[0,\infty)$, then $f \in L^1[0,\infty)$.

(iii) If $\{f_k\} \in L^p(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ for some $p, r \in [1,\infty)$, $f_k \to g$ in $L^p(\mathbb{R}^n)$ and $f_k \to h$ in $L^r(\mathbb{R}^n)$, then $g = h$ a.e. in $\mathbb{R}^n$.

2. Find or disprove:

(i) A sequence $\{f_k\}$ of functions in $L^1[0,\infty)$ such that $|f_k(x)| \leq 1$ for all x and for all k, $\lim_{k\to\infty} f_k(x) = 0$ for all x, and $\lim_{k\to\infty} \int_0^\infty f_k = 1$.

(ii) A sequence $\{f_k\}$ of Lebesgue measurable functions such that f_k converges to 0 in measure on $\mathbb{R}$, but no subsequence converges uniformly on any subset of positive measure.

3. Find the limit and justify your answer:

$$\lim_{n\to\infty} \int_0^{\pi/2} \sqrt{n \sin \frac{x}{n}}\, dx.$$

You can use the fact $0 \leq \sin\theta \leq \theta$ for $\theta \in [0,\pi/2]$.

4. (i) Show that

$$\int_1^\infty \frac{\sqrt[3]{1+x}}{x^2} dx \leq \sqrt[3]{6}.$$

(ii) Show that the strict inequality holds in (i).

5. Assume that $\{f_k\}$, $\{g_k\}$, f and g are in $L^1(\mathbb{R}^n)$, $f_k \to f$ pointwise a.e., $g_k \to g$ pointwise a.e., $|f_k| \le g_k$ a.e., and $\int_{\mathbb{R}^n} g_k\,dx \to \int_{\mathbb{R}^n} g\,dx$.

(i) Show that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k\,dx = \int_{\mathbb{R}^n} f\,dx.$$

(ii) Show that for every Lebesgue measurable set $E \subset \mathbb{R}^n$,

$$\lim_{k\to\infty} \int_E f_k\,dx = \int_E f\,dx.$$

6. Let $\varepsilon > 0$ and

$$f(x) = \begin{cases} x^{1+\varepsilon}\sin\frac{1}{x}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

Show that f is absolutely continuous on $[0,1]$.

7. Suppose f is real-valued and absolutely continuous on $[a,b]$. Show that

$$V(f,[a,b]) = \int_{[a,b]} |f'|\,dx.$$

Hint: Let $F(x) := V(f,[a,x])$, and show that $F+f$ and $F-f$ are absolutely continuous and nondecreasing. From this obtain a relationship between F' and f'.

8. Assume that $f \in L^1[0,\infty)$ and $\int_0^\infty f(x)\,dx = m$. Show that

(i) for a.e. $x \in [0,\infty)$, $F(x) := \int_0^x f(x-y)f(y)\,dy$ exists, and $F \in L^1[0,\infty)$;

(ii) $\lim_{n\to\infty} \int_0^n F(x)\,dx = m^2$.