Joint Program Exam, September 2005
Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem in PART 1, give a brief explanation that supports your answer. For each problem in PART 2 which you attempt to give a solution, the completeness is important: a correct and complete solution to one problem will gain more credit than two “half solutions” to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this test m and m_2 denote the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. ‘Measurable’ is short for ‘Lebesgue-measurable’. Instead of dm we sometimes write dx, refereeing to the variable to be integrated.

PART 1

DO ALL PROBLEMS IN PART ONE.

For each of following statements, determine whether it is true or false. Justify!

1. There is no sequence of measurable sets $\{E_n\}$ with the property $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ and
$$m(\cap_n E_n) \neq \lim_{n\to\infty} m(E_n)\,.$$

2. If $f : [0, \infty) \to \mathbb{R}$ is differentiable, then f' is measurable.

3. Let $E = [0, \infty)$. If $f \in L^1(E)$ and f is nonnegative, then $\lim_{x\to\infty} f(x) = 0$.

4. Let $E = [0, 1]$. If $f_n \in L^1(E)$ and $f_n \to 0$ pointwise as $n \to \infty$, then
$$\lim_{n\to\infty} \int_E f_n \, dm = 0.$$

5. If $f : [0, 1] \to \mathbb{R}$ is absolutely continuous, then f^2 is also absolutely continuous.

PART 2

DO 4 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED

1. Let E be a measurable subset of $\mathbb{R}$. Prove that there is a set F, which is a countable union of closed sets of $\mathbb{R}$, such that $F \subseteq E$ and $m(E \setminus F) = 0$. (Note: this is a standard theorem. You need to provide a complete proof of the theorem, not just re-state it.)

2. Let E be a measurable subset of $\mathbb{R}$. For a measurable function $f : E \to \mathbb{R}$, if $f \in L^p(E) \cap L^q(E)$, where $0 < p < q < \infty$, then $f \in L^r(E)$ for any $r \in (p, q)$.

3. Let $f \in L^1(\mathbb{R})$. Prove that
$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)|\, dx = 0.$$

4. Find the limit and justify your answer:
$$\lim_{n \to \infty} \int_1^\infty \frac{\ln(nx)}{x + x^2 \ln n}\, dx.$$

5. Let $I = [0,1] \times [0,1] \in \mathbb{R}^2$ and let $f : I \to \mathbb{R}$ be defined by
$$f(x,y) = \frac{1}{|x-y|^\alpha},$$
where $0 < \alpha < 1$. Prove that $f \in L^1(I)$ and find
$$\int_I f\, dm_2.$$

6. Suppose that $f : [a,b] \to \mathbb{R}$ is Lipschitz continuous. Prove that f' exists a.e in $[a,b]$, $f' \in L[a,b]$, and
$$f(x) = f(a) + \int_{[a,x]} f'\, dm \qquad \text{for all} \quad x \in [a,b].$$