University of Alabama System

Joint Ph.D Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May 2011

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. (a) Let $T \in \mathcal{L}(\mathbb{R}^n)$ be a linear map with $(T(\mathbf{x}), \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$, where $(\cdot, \cdot)$ denotes the inner-product on $\mathbb{R}^n$. Show that $T^* = -T$.

   (b) Let V and W be two vector spaces. Prove that if there exists a linear map $V \to W$ whose null space and range are both finite dimensional, then V is finite dimensional.

2. Let $\alpha = \{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$ be an arbitrary ordered basis for $\mathbb{R}^5$. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Define $T : \mathbb{R}^5 \to \mathbb{R}^5$ by $[T(\mathbf{v})]_\alpha = A[\mathbf{v}]_\alpha$ for all $\mathbf{v} \in \mathbb{R}^5$, where $[\mathbf{v}]_\alpha$ denotes the coordinate representation of $\mathbf{v}$ relative to the basis α.

   (a) Compute the eigenvalues of T and both the minimal and characteristic polynomial of T.

   (b) Find the Jordan form for T.

3. Let A be an $n \times n$ complex matrix. Define $M = \frac{1}{2}(A + A^*)$ and $N = \frac{1}{2}(A - A^*)$. Prove that A is normal if every eigenvector of M is also an eigenvector of N.

4. Let $A \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$ be a unit vector in the 2-norm, $\tau \in \mathbb{R}$ and $\mathbf{r} = A\mathbf{x} - \tau\mathbf{x}$.

   (a) Show that τ is an eigenvalue of a matrix $A + E$, where $\|E\|_2 \leq \|\mathbf{r}\|_2$.

   (b) Assuming in addition that A is symmetric, show that there exists an eigenvalue λ of A such that $|\lambda - \tau| \leq \|\mathbf{r}\|_2$.

5. Prove that any Givens rotator matrix in $\mathbb{R}^n$ is a product of two Householder reflector matrices. Can a Householder reflector matrix be a product of Givens rotator matrices?

6. Suppose an $m \times n$ matrix has the form $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where A_1 is a nonsingular matrix of dimension $n \times n$ and A_2 is an arbitrary matrix of dimension $(m-n) \times n$ with $m > n$. Let $A^\dagger$ be the pseudo inverse of A defined as $(A^*A)^{-1}A^*$. Prove that $\|A^\dagger\|_2 \leq \|A_1^{-1}\|_2$.

7. Let $A = U\Sigma V^T$ be a singular value decomposition of an $m \times n$ matrix. Let the nonzero singular values of A be $\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0$. Prove the following:

   (a) The rank(A) is r.

   (b) $\|A\|_2 = \sigma_1$, where $\|A\|_2$ is the $2-$ norm of A.

   (c) $\|A\|_F \leq \sqrt{\text{rank}(A)}\|A\|_2$, where $\|A\|_F$ is the Frobenius norm of A.

8. Let $S \in \mathbb{C}^{m\times m}$ be skew-Hermitian, i.e., $S^* = -S$. Prove the following:

   (a) The eigenvalues of S are purely imaginary.

   (b) The matrix $I - S$ is invertible.

   (c) The matrix $Q = (I - S)^{-1}(I + S)$ is unitary.