UNIVERSITY OF ALABAMA SYSTEM
Joint Doctoral Program in Applied Mathematics
Joint Program Exam: Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May 12 2005

Instructions: Do 7 of the 8 problems for full credit. Include all work. Write your student ID number on every page of your exam.

1. Let $A = I + x \cdot y^*$, where $x, y \in \mathbb{C}^m$ $(\neq 0)$ and I is the $m \times m$ identity matrix.

   (a) Determine a necessary and sufficient condition on x, y so that A admits an eigenvalue decomposition. Then find such a decomposition.

   (b) Determine a necessary and sufficient condition on x, y so that A admits an unitary diagonalization. Then find such a diagonalization.

2. Let $A, E \in \mathbb{R}^{m\times m}$ with $E \neq 0$ and $(A + E)$ being singular.

   (a) Prove
$$cond(A) \geq \|A\|/\|E\|$$
   for any matrix norm consistent with some vector norm.

   (b) Suppose A is non-singular and $\mathbf{y} \in \mathbb{R}^m$ is non-trivial satisfying
$$\|A^{-1}\|_2\|\mathbf{y}\|_2 = \|A^{-1}\mathbf{y}\|_2.$$
   Show that equality holds in the relation (a) for the 2-norm for
$$E = -\mathbf{y}\mathbf{x}^T/\|\mathbf{x}\|_2^2, \qquad \mathbf{x} = A^{-1}\mathbf{y}.$$

   (c) Use the inequality in (a) to get a lower bound for
$$cond_\infty(A) = \|A\|_\infty\|A^{-1}\|_\infty$$
   for the matrix
$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & \epsilon & \epsilon \\ 1 & \epsilon & \epsilon \end{pmatrix}$$
   where $0 < \epsilon < 1$.

3. Let $A \in \mathbb{R}^{n\times m}$ with $rank(A) = r \geq 0$.

   (a) Show that for every $\epsilon > 0$, there exists a full rank matrix $A_\epsilon \in \mathbb{R}^{n\times m}$ such that $\|A - A_\epsilon\| < \epsilon$.

   (b) Assume $r > 0$ and let $A = U\Sigma V^T$ be a SVD of A, with singular values $\sigma_1 \geq \sigma_2 \geq \cdot\cdot \geq \sigma_r > 0$. For each value $k = 0, 1, 2, \cdots, r-1$, define $A_k = U\Sigma_k V^T$ where Σ_k is the upper-left $k \times k$ sub-matrix of Σ. Show that
   (i) $\sigma_{k+1} = \|A - A_k\|_2$.
   (ii) $\sigma_{k+1} = \min\{\|A - B\|_2 : B \in \mathbb{R}^{n\times m} \text{ and } rank(B) \leq k\}$.

4. Let $A_1, A_2, \ldots, A_k \in F^{n\times n}$ such that A_1 has n distinct eigenvalues. Prove that there exists an invertible $P \in F^{n\times n}$ such that $P^{-1}A_jP$ is a diagonal matrix for each $1 \leq j \leq k$ if and only if $A_iA_j = A_jA_i$ for all $1 \leq i, j \leq k$.

5. (a) Let $x, y \in \mathbb{R}^n$ such that $x \neq y$ but $\|x\|_2 = \|y\|_2$. Show that there exists a reflector Q of the form $Q = I - 2uu^T$, where $u \in \mathbb{R}^n$ and $\|u\|_2 = 1$ such that $Qx = y$.

   (b) Let $A = \begin{bmatrix} 4 & 4 & 1 \\ 3 & -2 & 7 \\ 0 & 3 & 1 \end{bmatrix}$. Use the Householder reflector to find an QR factorization for the matrix A, i.e., $A = QR$ where Q is an orthogonal matrix and R is an upper triangular matrix.

6. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 3 \\ 1 & 0 \end{pmatrix}$.

   (a) Find an QR factorization of A by the Gram-Schmidt process.

   (b) Use the QR factorization from (a) to find the best least square fit by a linear function for $(1, -2)$, $(-2, 0)$, $(3, 2)$ and $(0, 3)$.

7. For which positive integers n does there exist $A \in \mathbb{R}^{n \times n}$ such that $A^2 + A + I = 0$. Justify your claim.

8. (In this problem, you may use Schur's factorization without proof).

   (a) Let $A \in \mathbb{C}^{m \times m}$. Show that A is normal (i.e., $AA^* = A^*A$) if and only if there is an unitary matrix V such that $A = A^*V$.

   (b) Assume that A is normal. Show that all eigenvalues of A are real if and only if A is hermitian.