UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 2004

Instructions: Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.

1. (a) Let $A \in \mathbb{C}^{n\times n}$ satisfy $A^* = -A$, where A^* is the conjugate transpose of A. Show that the matrix $I - A$ is invertible. Then show that the matrix $(I-A)^{-1}(I+A)$ is unitary.

   (b) Let $A \in \mathbb{C}^{n\times n}$. Prove that $\|Ax\|_2 = \|A\|_2\|x\|_2$ if and only if $A^*Ax = \lambda_{\max}x$, where $\lambda_{\max}$ is the largest eigenvalue of the matrix A^*A.

2. (a) Find a nonzero matrix $A \in \mathbb{R}^{2\times 2}$ that admits at least two LU decomposition, i.e. $A = L_1U_1 = L_2U_2$, where L_1 and L_2 are two *distinct* unit lower triangular matrices and U_1 and U_2 are two *distinct* upper triangular matrices.

   (b) Let

$$A = \begin{pmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 10 \\ -2 \\ -5 \end{pmatrix}.$$

   Use Gaussian elimination with partial pivoting to find matrices L and U such that U is upper triangular, L is lower triangular with $|l_{ij}| \le 1$ for all $i > j$ and $LU = \hat{A}$, where $\hat{A}$ can be obtained from A by interchanging rows. Use your LU decomposition to solve $A\mathbf{x} = \mathbf{b}$.

3. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. The corresponding *dual norm* $\|\cdot\|'$ is defined by the formula $\|x\|' = \sup_{\|y\|=1} |y^*x|$. Prove that the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are dual to each other. Prove that $\|\cdot\|$ coincides with $\|\cdot\|'$ if $\|\cdot\|$ is the 2-norm.

4. (a) Let $x, y \in \mathbb{C}^n$ be such that $x \neq y$ and $\|x\|_2 = \|y\|_2 \neq 0$. Show that there is a unique reflector matrix P such that $Px = y$ if and only if $\langle x, y\rangle \in \mathbb{R}$.

   (b) Prove or disprove: two matrices $A, B \in \mathbb{C}^{n\times n}$ are unitary equivalent if and only if they have the same singular values.

5. Let V be an n-dimensional vector space over $\mathbb{C}$. A linear operator T on V is said to be involutory if $T^{-1} = T$.

   (a) Prove that a linear operator on V is involutory if and only if it is diagonalizable with each eigenvalue equal to 1 or -1.

   (b) Suppose that $T = RS$ where R and S are involutory linear operators on V, and that the eigenvalues of T are $\lambda_1, \lambda_2, \cdots, \lambda_n$. Prove $\lambda_i \neq 0$ for $1 \le i \le n$, and

$$\{1/\lambda_1, 1/\lambda_2, \cdots, 1/\lambda_n\} = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}.$$

6. Let A be an $n \times n$ real matrix of full rank with eigenvalues $\lambda_1, \lambda_2, \ldots.., \lambda_n$ and let X be a matrix that diagonalizes A, i.e. $X^{-1}AX = D$ where D is a diagonal matrix. If $A' = A + E$ and λ' is an eigenvalue of A', prove that

$$\min_{1 \le i \le n} |\lambda' - \lambda_i| \le \kappa_2(X)\|E\|_2$$

where $\kappa_2(X)$ is the 2-norm condition number of X.

7. Recall that a matrix A is normal if $AA^* = A^*A$. Prove

   (a) If A is a normal matrix then A and A^* have the same eigenvectors.

   (b) If A is a normal matrix and two vectors x and y are eigenvectors of A corresponding to different eigenvalues, then the vectors x and y are orthogonal.

   (c) If A is a normal and upper triangular matrix then A is diagonal.

8. Given the data $(0, 1), (2, 4), (5, 6)$. Use a QR factorization technique to find the best least squares fit by a linear function.