UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 2003

Instructions: Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.

1. (a) Let $A \in \mathbb{C}^{n\times n}$ be unitary, Hermitian and positive definite. Show that $A = I$.

   (b) Prove that if H is Hermitian and nonnegative semidefinite then there exists a Hermitian nonnegative semidefinite matrix G such that $G^2 = H$.

2. The spectral radius of $A \in \mathbb{C}^{n\times n}$ is defined by

$$\rho(A) = \max\{|\lambda| : \ \lambda \text{ an eigenvalue of } A\}.$$

   Show that

   (a) $\rho(A) \leq ||A||$ for every matrix norm $||\cdot||$ that is induced by a norm on $\mathbb{C}^n$.

   (b) if A is normal then $||A|| = \rho(A)$, when $\|\cdot\|$ is induced by $\|\cdot\|_2$ on $\mathbb{C}^n$.

3. Consider the system

$$\begin{pmatrix} \varepsilon & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   Assume that $|\varepsilon| \ll 1$. Solve the system by using the LU decomposition with and without partial pivoting and adopting the following rounding off model (at all stages of the computation!):

$$a + b\varepsilon = a \quad (\text{for } a \neq 0),$$

$$a + b/\varepsilon = b/\varepsilon \quad (\text{for } b \neq 0)).$$

   Find the exact solution, compare and make comments.

4. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$.

   (a) Let AB have minimal polynomial $m_1(x)$, and let BA have minimal polynomial $m_2(x)$. Prove that one of the following holds: $m_1(x) = m_2(x)$, $\ m_1(x) = xm_2(x)$, or $m_2(x) = xm_1(x)$,

   (b) Let AB be nonsingular. Prove that AB is diagonalizable if and only if BA is diagonalizable.

5. (a) Given $\mathbf{x} = (2, 2, 1)^T$, find an orthogonal matrix Q such that $Q\mathbf{x}$ is parallel to $\mathbf{e}_1 = (1, 0, 0)^T$.

   (b) Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$, where

$$A = \begin{pmatrix} 3 & 1 & 2 \\ -4 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}.$$

6. For any $X = (x_{ij}), Y = (y_{ij}) \in \mathbb{R}^{n\times n}$, define

$$(X, Y) = \sum_{i,j=1}^{n} x_{ij} y_{ij}.$$

(a) Prove that $(\cdot, \cdot)$ is an inner product on $\mathbb{R}^{n\times n}$.

(b) Given $A \in \mathbb{R}^{n\times n}$, define $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $L(X) = AX, \quad X \in \mathbb{R}^{n\times n}$. Show that L is a linear operator and determine its adjoint with respect to the inner product defined in part (a).

7. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues such that $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| > |\lambda_n|$. Suppose $\mathbf{z} \in \mathbb{R}^n$ with $\mathbf{z}^T \mathbf{x}_1 \neq 0$, where $A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$. Prove that, for some constant C,

$$\lim_{k\to\infty} \frac{A^k \mathbf{z}}{\lambda_1^k} = C\mathbf{x}_1$$

and use this result to devise a reliable algorithm for computing λ_1 and $\mathbf{x}_1$. Explain how the calculation should be modified to obtain (a) λ_n and (b) the eigenvalue closest to 2.

8. Consider the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}.$$

(a) Determine the singular value decomposition (SVD) of A in the form $A = U\Sigma V^T$.

(b) Compute the condition number $\kappa_2(A^T A)$ using the SVD of A.