

UNIVERSITY OF ALABAMA SYSTEM  
JOINT DOCTORAL PROGRAM IN APPLIED  
MATHEMATICS

JOINT PROGRAM EXAMINATION  
Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 2000

**Instructions:** Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.

1. Let  $V$  be a vector space of finite dimension  $n$ , let  $T$  be a linear operator on  $V$  with  $k + 1$  distinct eigenvalues  $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_k$ , and let the eigenspace corresponding to  $\lambda_0$  have dimension  $n - k$ . Prove that the operator  $T^m$  is diagonalizable for each positive integer  $m$ .
2. Let  $V$  and  $W$  be finite dimensional vector spaces, and let  $T : V \rightarrow W$  be a linear transformation of rank  $r$  where  $1 \leq r < \min\{\dim(V), \dim(W)\}$ . Prove that there exist bases  $\alpha$  for  $V$  and  $\beta$  for  $W$  such that the matrix representation for  $T$  with respect to  $\alpha$  and  $\beta$  has the form  $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ .
3. Let  $V$  be a finite dimensional vector space with inner product  $\langle \cdot, \cdot \rangle$ , and let  $T$  be a self-adjoint operator on  $V$ . Prove that there exists a self-adjoint operator  $S$  on  $V$  such that  $T = S^2$  if and only if  $\langle T\mathbf{x}, \mathbf{x} \rangle \geq 0$  for all  $\mathbf{x} \in V$ .
4. (a) Let  $A$  be a  $10 \times 10$  complex matrix with characteristic polynomial  $C_A(x) = (x - 1)^6(x + 2)^4$ , minimal polynomial  $M_A(x) = (x - 1)^3(x + 2)^2$ , and  $\dim E_1 = 3$ ,  $\dim E_{-2} = 2$ , where  $E_1$  and  $E_{-2}$  are the eigenspaces corresponding to the eigenvalues 1 and  $-2$  respectively. Find a Jordan canonical form of  $A$ .  
 (b) Let  $A$  be an  $8 \times 8$  complex matrix with characteristic polynomial  $C_A(x) = (x + i)^3(x - i)^3(x - 1)^2$ , and  $\dim E_{-i} = \dim E_i = \dim E_1 = 2$ . Find the minimal polynomial of  $A$ .
5. (a) Calculate  $A^{-1}$  and  $\kappa_\infty(A)$  for the matrix

$$A = \begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix}.$$

- (b) For the above  $A$ , find  $\mathbf{b}$ ,  $\delta\mathbf{b}$  and  $\mathbf{x}$ ,  $\delta\mathbf{x}$  such that

$$A\mathbf{x} = \mathbf{b}, \quad A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$$

with  $\|\delta\mathbf{b}\|_\infty / \|\mathbf{b}\|_\infty$  small and  $\|\delta\mathbf{x}\|_\infty / \|\mathbf{x}\|_\infty$  large.

- (c) Let  $A \in \mathbb{R}^{n \times n}$  be given, nonsingular, and consider the linear system problem

$$A\mathbf{x} = \mathbf{b},$$

where  $\mathbf{b} \in \mathbb{R}^n$  is given. Let  $\mathbf{x} + \delta\mathbf{x} \in \mathbb{R}^n$  be an approximate solution to this problem, satisfying

$$A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}.$$

Prove that

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A) \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

and comment on the significance of this result.

6. Use a  $QR$  decomposition, with exact arithmetic, to solve the least squares problem for the overdetermined system

$$\begin{bmatrix} 1 & -3 \\ 2 & 4 \\ 2 & 5 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix}.$$

State the magnitude of the minimum residual.

7. Let  $A \in \mathbb{R}^{n \times n}$  be given, and let  $Q_0$  be an arbitrary  $n \times n$  orthogonal matrix. Consider the sequence of matrices  $R_k$  and  $Q_k$  computed as follows:

$$\begin{aligned} Z_{k+1} &= AQ_k, \\ Q_{k+1}R_{k+1} &= Z_{k+1}. \end{aligned}$$

In the last step, we compute the  $QR$  decomposition of  $Z_{k+1}$  to get  $Q_{k+1}$  and  $R_{k+1}$ . Assume that

$$\lim_{k \rightarrow \infty} Q_k = Q_\infty$$

and

$$\lim_{k \rightarrow \infty} R_k = R_\infty$$

exist. Prove that the eigenvalues of  $A$  are given by the diagonal elements of  $R_\infty$ .

8. Let  $A \in \mathbb{C}^{n \times n}$  be given, Hermitian, and let  $(\lambda, u)$  be an arbitrary eigenpair of  $A$ , with  $u$  real and  $\|u\|_2 = 1$ . Let  $x \approx u$  be given, with  $\|x\|_2 = 1$  and define  $\sigma$  by

$$\sigma = \frac{(Ax, x)}{(x, x)},$$

the Rayleigh quotient of  $x$ . Prove that

$$|\lambda - \sigma| \leq C\|u - x\|_2^2.$$