UNIVERSITY OF ALABAMA SYSTEM
Joint Doctoral Program in Applied Mathematics
Joint Program Exam: Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September 2006

Instructions: Do 7 of the 8 problems for full credit. Include all work. Write your student ID number, and problem number, on every page.

1. Let

$$A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{pmatrix}.$$

   (a) Determine conditions on a, b, c, and d so that there is only one Jordan block for each eigenvalue of A in the Jordan canonical form of A.

   (b) Suppose now $a = c = d = 2$ and $b = -2$. Find the Jordan canonical form of A.

2. Let

$$A = \begin{pmatrix} -1 & 4 & -2 \\ -2 & 5 & -2 \\ -1 & 2 & 0 \end{pmatrix}$$

   with characteristic polynomial $\Delta(x) = (x-1)^2(x-2)$.

   (a) For each eigenvalue λ of A find a basis for the eigenspace E_λ.

   (b) Determine if A is diagonalizable. If so, give matrices P, B such that $P^{-1}AP = B$ and B is diagonal. If not, explain carefully *why* A is not diagonalizable.

3. Let V be a vector space over a field F and let W_1 and W_2 be finite dimensional subspaces of V. Prove that $W_1 \cap W_2$ and $W_1 + W_2 = \{u + v : u \in W_1, v \in W_2\}$ are finite-dimensional subspaces of V, and that

$$\dim(W_1 \cap W_2) + \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2).$$

4. Let $A = (a_{kj}) \in \mathbb{R}^{n \times n}$ be symmetric and positive definite. Show that

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1n} & x_1 \\ & \cdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & x_n \\ x_1 & \cdots & x_n & 0 \end{pmatrix} < 0$$

   for every nonzero vector $x = (x_1, \ldots, x_n)^t$.

5. Consider the system

$$\begin{pmatrix} \varepsilon & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   Assume that $|\varepsilon| \ll 1$.

(a) Solve the system by using the LU decomposition with partial pivoting and adopting the following rounding off models (at all stages of the computation!):

$$a + b\varepsilon = a \quad \text{for } a \neq 0,$$

and

$$a + b/\varepsilon = b/\varepsilon \quad \text{for } b \neq 0.$$

(b) Solve the system by using the LU decomposition without partial pivoting and adopting the same rounding off models as in (a).

(c) Find the exact solution, compare, and make comments.

6. (a) Let $x, y \in \mathbb{R}^n$ be such that $x \neq y$ and $||x||_2 = ||y||_2 \neq 0$. Show that there is a unique reflection matrix P such that $Px = y$.

   (b) Let $x, y \in \mathbb{C}^n$ be such that $x \neq y, ||x||_2 = ||y||_2 \neq 0$ and the Euclidean inner product of x and y is real. Show that there is a reflection matrix Q such that $Qx = y$.

7. (a) Find the reduced QR factorization of $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ 2 & -5 \end{bmatrix}$.

   (b) Use the result in part (a) to find

      i. the least squares solution of the system $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -11 \\ -11 \end{bmatrix}$ and the corresponding residual vector; and

      ii. the orthogonal projector on the column space of A (without using A itself, but in terms only of the orthogonal factor of A).

8. (a) Given $A \in \mathbb{R}^{m\times n}$, show that the nonzero eigenvalues are the same for A^TA and AA^T.

   (b) For the matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1/2 & \sqrt{3}/2 \\ \sqrt{3} & 1 \end{bmatrix}$, obtain the singular value decomposition of A (in the form $A = U\Sigma V^T$ where U and V are orthogonal matrices).