UNIVERSITY OF ALABAMA SYSTEM
Joint Doctoral Program in Applied Mathematics
Joint Program Exam: Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September 15 2005

Instructions: Do 7 of the 8 problems for full credit. Include all work. Write your student ID number on every page of your exam.

1. Let V be the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ which have degree less than or equal to $n-1$. Let $t_1, t_2, \cdots, t_n$,be any n distinct real numbers, and define linear functionals $L_i(p) = p(t_i)$ on V. Show that $L_1,\ L_2, \cdots, L_n$ are linearly independent.

2. For $C \in F^{m\times m}$, define $T_C(X) = CX$ for all $X \in F^{m\times n}$. Prove the following.

   (a) The function T_C is a linear operator on the vector space $F^{m\times n}$.

   (b) If $\text{rank}(C) = r$, then $\text{rank}(T_C) = nr$.

   (c) The minimal polynomial of T_C is the same as the minimal polynomial of C.

   (d) The linear operator T_C is diagonalizable if and only if C is diagonalizable.

3. Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Prove or disprove that the non-zero eigenvalues of $AB - BA$ are pure imaginary.

4. Let V be an n-dimensional vector space over field of complex numbers $\mathbb{C}$, and let I be the identity operator on V. Prove: For linear operators S and T on V, there exist ordered bases α and β for V such that the matrix representation of S with respect to α is equal to the matrix representation of T with respect to β if and only if $\text{rank}(cI + S)^k = \text{rank}(cI + T)^k$ for all $c \in \mathbb{C}$ and $k = 1, 2, \cdots, n$.

5. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Show that $\min\{\frac{\|\delta A\|_2}{\|A\|_2} \mid A+\delta A \text{ is singular}\} = 1/\kappa_2(A)$. (That is the relative distance to the nearest singular matrix is $1/\kappa_2(A)$.)

6. For any matrix $A = (a_{ij})$, $A \in \mathbb{R}^{n\times m}$ define

$$||A||_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2\right)^{1/2}$$

   This is the Frobenius matrix norm. Show that if $Q \in \mathbb{R}^{n\times n}$ is orthogonal, then $||QA||_F = ||A||_F$. Then show that

$$||A||_F = (\sigma_1^2 + \cdots + \sigma_r^2)^{1/2}$$

   where σ_i are singular values of A.

7. Consider the linear least squares (LS) problem

$$\min_x \|b - Ax\|_2, \quad A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \; b = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}. \qquad (1)$$

   Compute the QR decomposition of A using the Householder reflections and then solve the LS problem (1) using the QR decomposition method.

8. Let $A, Q_0 \in \mathbb{R}^{m\times m}$. Define sequences of matrices Z_k, Q_k and R_k by

$$Z_k = AQ_{k-1}, \quad Q_k R_k = Z_k, \qquad k = 1, 2, \cdots ,$$

where $Q_k R_k$ is an QR factorization of Z_k. Suppose $\lim_{k\to\infty} R_k = R_\infty$ exists, and $\lim_{k\to\infty} Q_k = Q_\infty$ exist. Determine the eigenvalues of A in terms of R_∞.