# UNIVERSITY OF ALABAMA SYSTEM
# JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
# JOINT PROGRAM EXAMINATION
## Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September 16, 2004

**Instructions:** Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.

1. Let $V$ and $W$ be vector spaces over $F$, and let $S : V \to W$ and $T : W \to V$ be linear transformations. Prove the following.

   (a) If $\lambda$ is a nonzero eigenvalue of $TS$ then $\lambda$ is also an eigenvalue of $ST$.

   (b) Zero being an eigenvalue of $TS$ does not imply that zero is an eigenvalue of $ST$.

2. Suppose the 2-condition number of a rectangular matrix $A \in \mathbb{R}^{m\times n}$ is defined by

$$\kappa_2(A) : = \frac{\sup_{\|x\|_2=1} \|Ax\|_2}{\inf_{\|x\|_2=1} \|Ax\|_2}.$$

   Prove that $(\kappa_2(A))^2 = \kappa_2(A^T A)$.

3. Compute the LU decomposition $A = LU$ for the matrix

$$A = \begin{bmatrix} 0.01 & 2 \\ 1 & 3 \end{bmatrix}.$$

   Compute $\|L\|_\infty \|U\|_\infty$. What does this imply about the numerical stability of solving a system of linear equations $Ax = y$ by LU decomposition without pivoting?

4. Apply the QR algorithm (without shift) to the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   Does it converge and produce the eigenvalues of $A$? Explain why. Apply the QR algorithm with the Rayleigh quotient shift. Does it help the convergence? Explain why.

5. Suppose that $A$ is normal; i. e. $AA^H = A^H A$. Show that if $A$ is also triangular, it must be diagonal. Use this to show that an $n \times n$ matrix is normal if and only if it has $n$ orthonormal eigenvectors. (*Hint:* Show that $A$ is normal if and only if its Schur form is normal.)

6. Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of $A$, and $A$ be diagonalizable such that $X^{-1}AX = D = diag(\lambda_1, \cdots, \lambda_n)$. Prove that if $A' = A + E$ and $\lambda'$ is an eigenvalue of $A'$, then

$$\min_{1\le i\le n} |\lambda' - \lambda_i| \le \kappa_\infty(X)\|E\|_\infty.$$

7. Let $A = I + xy^T$, where $x$ and $y$ are nonzero n-vectors. Show that $det(A) = 1 + x^T y$. Determine the Jordan canonical form of the matrix A.

8. Let $T : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ be the transformation defined by $T(A) = (A + A^T)/2$.

   (a) Prove that $T$ is linear.

   (b) Find a basis of the null space of $T$ and determine its dimension