STUDENT SUCCESS CENTER
THE UNIVERSITY OF ALABAMA IN HUNTSVILLE

# CAL C: POLAR, CYLINDRICAL, & SPHERICAL COORDINATES

## Polar and Cylindrical Coordinates:

*Jacobian of the transformation*

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

- For limits of integration for a polar double integral, draw a ray extending radially outward from the origin and determine the functions at which it enters and at which it exits. For the $\theta$ limits, determine the initial and final angle of this ray. *Note: $\theta$ will usually be the last variable of integration.*

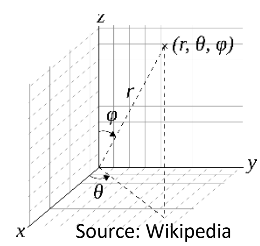
- The Cylindrical coordinate system is built on the polar coordinate system with the addition of a variable to describe the distance from the $r$-$\theta$ plane (thought of as height"), notated as $z$. Note that this $z$ is equivalent to that of the rectangular coordinate $z$.

*Jacobian of the transformation*

$$\iiint_D f\,dV = \iiint_D f\,dz\,r\,dr\,d\theta$$

- The limits of integration are similar to polar for $r$ and $\theta$ and to rectangular for $z$. *Note: $\theta$ will usually be the last variable of integration.*

## Spherical Coordinates:



Source: Wikipedia

- Spherical coordinates are defined by three parameters:
  1) $\rho$, the radial distance from a point to the origin.
  2) $\phi$, the polar angle between a point and the positive z-axis.
  3) $\theta$, the azimuth angle between the shadow of $\rho$ on the x-y plane and positive x-axis.

*Jacobian of the transformation*

$$\iiint_D f(x,y,z)dV = \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)dV = \iiint_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

- The limits of integration are found by similar fashion as cylindrical. Typically, first find $\rho$ limits by drawing a ray from the origin through the region at angle $\phi$ and observe the functions which it enters and exits. Next, determine $\phi$ limits by observing the minimum ($\leq 0°$) and maximum ($\geq 180°$) angle made with the positive z-axis. Finally, observe the "shadow" of ray $\rho$ on the x-y plane and determine the minimum and maximum angles it makes with the positive x-axis as it sweeps through the entire region.

| Cylindrical to Rectangular | Spherical to Rectangular | Spherical to Cylindrical |
|---|---|---|
| $x = r\cos\theta$ | $x = \rho\sin\phi\cos\theta$ | $r = \rho\sin\phi$ |
| $y = r\sin\theta$ | $y = \rho\sin\phi\sin\theta$ | $z = \rho\cos\phi$ |
| $z = z$ | $z = \rho\cos\phi$ | $\theta = \theta$ |

| | | |
|---|---|---|
| $dV =$ | $dx\,dy\,dz$ | Rectangular |

| | | |
|---|---|---|
| | $dz\, r\, dr\, d\theta$ | Cylindrical |
| | $\rho^2 sin\phi\, d\rho\, d\phi\, d\theta$ | Spherical |

## Change of Variables:

- A change of variables or coordinate systems is often useful in solving complex integrals. When doing so, a Jacobian is a necessary accompaniment:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v} = \frac{\partial(x,y)}{\partial(u,v)}$$

- For example, the following Jacobian is required to transform cartesian to polar coordinates:

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial(rcos\theta)}{\partial r} & \frac{\partial(rcos\theta)}{\partial \theta} \\ \frac{\partial(rsin\theta)}{\partial \theta} & \frac{\partial(rsin\theta)}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

- Likewise, it can be shown that $J(\rho,\phi,\theta) = \rho^2\, sin\phi$ by adding a third row $\left(\frac{\partial z}{\partial()}\right)$ and third column $\left(\frac{\partial()}{\partial \phi}\right)$.