

MEASUREMENT OF THE DISPERSION OF RADIATION FROM A STEADY COSMOLOGICAL SOURCE

RICHARD LIEU¹, LINGZE DUAN¹, AND T. W. B. KIBBLE²

¹ Department of Physics, University of Alabama, Huntsville, AL 35899, USA

² Blackett Laboratory, Imperial College, London SW7 2AZ, UK

Received 2013 June 13; accepted 2013 September 23; published 2013 November 6

ABSTRACT

The “missing baryons” of the near universe are believed to be principally in a partially ionized state. Although passing electromagnetic waves are dispersed by the plasma, the effect has hitherto not been utilized as a means of detection because it is generally believed that a successful observation requires the background source to be highly variable, i.e., the class of sources that could potentially deliver a verdict is limited. We argue in two stages that this condition is not necessary. First, by modeling the fluctuations on macroscopic scales as interference between wave packets, we show that, in accordance with the ideas advanced by Einstein in 1917, both the behavior of photons as bosons (i.e., the intensity variance has contributions from Poisson and phase noise) and the van-Cittert-Zernike theorem are a consequence of wave-particle duality. Nevertheless, we then point out that, in general, the variance on some macroscopic timescale τ consists of (1) a main contributing term $\propto 1/\tau$, plus (2) a small negative term $\propto 1/\tau^2$ due to the finite size of the wave packets. If the radiation passes through a dispersive medium, this size will be enlarged well beyond its vacuum minimum value of $\Delta t \approx 1/\Delta\nu$, leading to a more negative (2) term (while (1) remains unchanged), and hence a suppression of the variance wrt the vacuum scenario. The phenomenon, which is typically at a few parts in 10^5 level, enables one to measure cosmological dispersion in principle. Signal-to-noise estimates, along with systematic issues and how to overcome them, will be presented.

Key words: cosmological parameters – plasmas – radiation mechanisms: general

1. INTRODUCTION

The whereabouts of baryons in the low redshift universe remains a major unsolved problem of cosmology. While at higher redshifts of $z \gtrsim 2$, observations of the Ly α Forest and the Gunn–Peterson trough (Rauch 1998; Becker et al. 2001) reveal an adequate amount of baryons, the marked $\sim 50\%$ deficit at $z \lesssim 1$ between the observed density and that expected from nucleosynthesis and the standard cosmological model (Burles & Tytler 1998; Ade et al. 2013; Komatsu et al. 2011) was pointed out previously by Fukugita et al. (1998). Contemporaneously, two other papers, Cen & Ostriker (1999) and Davé et al. (2001), conjectured by cosmological hydrodynamic simulations that the missing baryons in question may have taken refuge in the temperature and density regimes of 10^5 – 10^7 K and 10^{-4} – 10^{-6} cm⁻³, as “warm” intergalactic gas in this part of parameter space is difficult to trace for many reasons (such as Galactic absorption of the emitted radiation or the gas may have an unusual composition or ionization state).

Nevertheless, because the models predicted a convergence of warm “filaments” onto the outskirts of clusters of galaxies, the largest bound systems in the universe that were formed from the collapse of primordial density fluctuations (Press & Schechter 1974; White & Rees 1978), there has been an intensified search campaign with nearby clusters as the focal point. The bulk of the X-ray and Sunyaev–Zeldovich measurements to date (Vikhlinin et al. 2006; Afshordi et al. 2007; Arnaud et al. 2007; Sun et al. 2009; Ettori et al. 2009), with the notable exceptions of Landry et al. (2013) and Simionescu et al. (2011), reveal that the mass ratio of baryons-to-total matter falls below the value predicted by the standard model, thereby providing an “inside out” perspective to the existence of baryons beyond the cluster’s virial radius. In addition to being “warm,” most recently, hints that dark matter is self-interacting, and hence exhibiting “baryonic” characteristics, are pointed out by e.g., Zavala et al. (2013). One way or another, it is clear that the “missing baryon” syndrome

persists, and the evidence uncovered has been circumstantial at best, with the only report of a direct detection thus far being the soft X-ray and EUV “excess” emission at $\sim 10^6$ K from cluster halos (Mittaz et al. 1998; Bonamente et al. 2002).

Turning to the question of observational techniques, since most of the baryons at $z \lesssim 1$ (and even higher z) are partially ionized, any electromagnetic radiation passing through the intergalactic medium is expected to be dispersed by them, to varying degrees, depending upon the radiation frequency. Indeed, the measurement of dispersion of radiation from a distant source would provide the most reliable means of determining the line-of-sight column density of all ionized gas components. However, it is generally thought (but not proven) that for the method to succeed there must be a “benchmark,” or reference, signal to enable one to measure the differential dispersive delay of the various Fourier components constituting the distant emission, and this will exist only if the source is periodic and fast, i.e., a pulsar.

Indeed, while pulsars are a powerful tool for mapping the distribution of Galactic baryons, the fact that they have hitherto not been detected at cosmological separations means that nothing can be done to scales beyond the Local Group. In fact, on such distance scales, the majority of the sources seen are quasars, which are, for this purpose, steady sources. Note that the very exciting results on fast radio bursts of possible extragalactic origin, viz. the confirmation by Thornton et al. (2013) of the earlier findings of Lorimer et al. (2007), may represent a change of the observational situation. Nevertheless, the absence of redshifts from this new class of sources renders it difficult to convert dispersive column densities of their intervening intergalactic media (IGM) into average volume densities out to some specific depths; see McQuinn (2013). Even if a means of ascertaining distances is in hand, a detectability analysis by Lorimer et al. (2013) indicates that such sources may only be seen out to redshifts of $z \leq 0.5$ by surveys below 1 GHz. There is also the ambiguity introduced by intrinsic dispersion

at the source itself, as pointed out by McQuinn (2013), which is not yet well understood for radio transients. Quasars, on the other hand, have been extensively studied from this viewpoint. As can be seen in, e.g., Figure 1 of Elvis (2002), $\approx 40\%$ – 45% of AGNs show no evidence of warm ionizing absorbers. Since from McKernan et al. (2007) the lowest detectable ionizing (H II) column of warm absorbers is $\approx 10^{20} \text{ cm}^{-2}$, which is well below typical IGM values even to nearby quasars, this means that a good fraction of quasars can be reliable probes of the IGM baryons if there is a way of inferring line-of-sight dispersion to such relatively steady sources.

In spite of all the above, we argued in a recent paper (Lieu & Duan 2013) that even the electromagnetic radiation of a steady source could carry imprint signatures of dispersion in a way that enables one to also recover the total line-of-sight column density of the ionized baryons. The imprint is embedded and can be uncovered only after a careful analysis of the statistical properties of the fluctuations in the arriving radiation, which are slightly different from normal undispersed radiation. In this paper, we intend to discuss in detail the difference and how one might detect it.

The premise of our work is the observation of Einstein (1917), developed subsequently by Purcell (1956), Hanbury-Brown & Twiss (1957), and Mandel (1958), that the variance of photon number fluctuations comprises Poisson and phase noise components, due, respectively, to the particle and wave nature of radiation. Indeed, as explained in Lieu & Duan (2013), the wave packet approach to radiation can account for not only the coherence length, or spatial extent, of interference fringe patterns (viz. the van Cittert-Zernike theorem, see, e.g., Section 10.4.2 of Born & Wolf (1970)) even in the single photon limit, but also why this length is not affected by dispersion. Here we will show by modeling photons semi-classically as wave packets that the fluctuations of chaotic light from a steady source are fully consistent with Bose–Einstein statistics, i.e., our model is robust as it yields the same results as those from a formal quantum description of light. In the course of the analysis, it will become apparent that there exists a higher order distortion of the variance due to the finite size of the wave packet. The effect is more severe when dispersion by the medium of propagation stretches this size, and herein lies the imprint of the medium. Since we ignored quantum vacuum corrections, it must be emphasized that the approach we adopted is valid in the classical domain when the time and length scales of interest far exceed the wave packet.

2. THE SOURCE

Consider first the radiation at the source, assumed to be at $x = 0$. Let us suppose that it comprises a sequence of wave packets, each of the form

$$\Phi(t, 0) = \sum_j g(t - t_j) e^{i\varphi_j}, \quad (1)$$

where the emission times t_j are randomly distributed with a constant rate of λ pulses per unit time, and the phases φ_j randomly distributed between 0 and 2π . Here λ may also be interpreted as the number of photons arriving at the detector per unit time. In other words, during a long interval t_{exp} of exposure to the source, the expected number of photons entering the detector is λt_{exp} . However, there is no one-to-one correspondence between the two sides, i.e., one cannot say

which of the photons emitted corresponds to which of those detected.

Next, we can reasonably assume that $\int dt g(t) = 0$. Let us define

$$G(t) = \int dt' g(t') g^*(t - t'), \quad (2)$$

or equivalently in terms of Fourier transforms

$$\tilde{G}(\omega) = \int dt G(t) e^{i\omega t} = |g(\omega)|^2. \quad (3)$$

Since the rate λ is uniform, averages such as $\langle \Phi(t) \Phi^*(t') \rangle$ or $\langle I(t) I(t') \rangle$ where $I(t) = |\Phi(t)|^2$ are functions only of the time difference $t - t'$. Explicitly, because the emission times t_j are uncorrelated, we need only consider correlations of each wave packet with itself, so

$$\langle \Phi(t, 0) \Phi^*(t', 0) \rangle = \lambda G(t - t'). \quad (4)$$

That is equivalent to saying that the Fourier components of Φ with different frequencies are uncorrelated:

$$\langle \tilde{\Phi}(\omega) \tilde{\Phi}^*(\omega') \rangle = 2\pi \lambda \delta(\omega - \omega') \tilde{G}(\omega), \quad (5)$$

where $\langle X \rangle$ refers to the average value of X over a time period much longer than the duration of the wave packets. In particular, the relation

$$\langle I(t) \rangle = \lambda G(0) \quad (6)$$

is obtained by setting $t = t'$.

We turn to the intensity correlation function. Here we have a product of four fields, each of which is a sum, as in Equation (1). Any contribution where one t_j appears in only a single factor will always vanish. There are therefore two distinct types of contribution we need to consider. First, we have those from pairwise correlations in which two photons j, k ($j \neq k$) contribute. In the limit of large λ (i.e., $\lambda \gg \delta\nu$, the width of the distribution \tilde{G}) where there is strong overlap between wave packets, this is the dominant contribution. It is

$$\langle I(t) I(t') \rangle = \lambda^2 [G(0)^2 + |G(t - t')|^2], \quad (7)$$

or equivalently,

$$\langle I(t) I(t') \rangle - \langle I \rangle^2 = \lambda^2 |G(t - t')|^2. \quad (8)$$

So in the large- λ limit, we have

$$\langle I^2(t) \rangle = 2\lambda^2 |G(0)|^2 = 2\langle I \rangle^2, \quad (9)$$

and the standard deviation in the intensity equals the mean intensity (if the source is exposed to the observer for a total time t_{exp} , so that the total number of photons is some large number λt_{exp} , then, strictly speaking, the omission of the $j = k$ term would mean replacing $\lambda^2 t_{\text{exp}}^2$ by $\lambda t_{\text{exp}} (\lambda t_{\text{exp}} - 1)$, but that correction is negligible if $\lambda t_{\text{exp}} \gg 1$). Note also that the mean value $\langle \Phi(t) \Phi(t') \rangle$ will vanish when the random phases are taken into account.

The second type of contribution comes from the case $j = k$ where all four wave packets are the same. That term is the dominant one in the opposite limit of small λ . It is

$$\langle I(t) I(t') \rangle \approx \lambda \int dt'' |g(t + t'')|^2 |g(t' + t'')|^2. \quad (10)$$

In either case, the correlation function depends only on the time difference and will tend to zero when $t - t'$ is much larger than the width of the function $G(t)$ (or of $g(t)$).

3. A DISTANT OBSERVER AND GAUSSIAN WAVE PACKETS

We treat the propagation of radiation from an unresolvable point source to a small telescope as principally a one-dimensional problem $\mathbf{k} = (k, 0, 0)$ by ignoring the dynamics of the wave packet in the y and z directions. Although dispersion broadens the wave packet predominantly along x , another effect—scattering—does so in all directions. However, as will be discussed in the end of Section 4, for the purpose of this paper, scattering is negligible in the radio and even more so at shorter wavelengths.

Let the observer be located at some distant point x . Clearly, for a uniform medium of propagation,

$$\Phi(t, x) = \int \frac{d\omega}{2\pi} \tilde{\Phi}(\omega) e^{-i\omega t + ik(\omega)x}. \quad (11)$$

This is equivalent to saying that the photon pulse shape function $\tilde{g}(\omega)$ is modified by the effect of dispersion

$$\tilde{g}(\omega) \rightarrow \tilde{g}_x(\omega) = \tilde{g}(\omega) e^{i[k(\omega) - \omega/c]x}, \quad (12)$$

where the final factor allows for the time development in the absence of dispersion. Note that from Equation (3), this means that $G(t)$ is unaffected by dispersion. It immediately follows that there is no change in the correlation function of Φ , given by Equation (4). Hence, the coherence length of the radiation, defined as the spatial extent of the interference pattern and not the size of the wave packet (see the end of Section 1), is invariant wrt dispersion, as is the large- λ contribution to the intensity correlation function given by Equation (7) or (8). For more elaboration on the coherence length and its invariance, see the end of Section 2 of Lieu & Duan (2013). We shall return to the effect of dispersion on the small- λ contribution (Equation (10)) below.

To proceed further, it is convenient to choose a particular form for the function g . Specifically, we shall assume that it is a monochromatic beam modulated by a Gaussian envelope function, i.e., a superposition of Fourier components spanning the frequency range $\delta\nu$ at ν

$$g(t) = a e^{-i\omega_0 t} e^{-t^2/2(\delta t)^2}, \quad (13)$$

where a , $\omega_0 = 2\pi\nu$, and

$$\delta t = \frac{1}{\sqrt{2}\delta\omega} = \frac{1}{2\sqrt{2}\pi\delta\nu} \quad (14)$$

are constants. It follows that

$$\tilde{g}(\omega) = \sqrt{2\pi} a \delta t e^{-[\delta t(\omega - \omega_0)]^2/2}. \quad (15)$$

In this way, a detected photon of chaotic light is treated classically as a pulse emitted by the source which lasts about the reciprocal of the larger of the two bandwidths, the receiver's and the source's. Dispersive propagation does not alter the pulse spectrum but can lengthen the pulse duration after emission. The fact that we took $1/\delta\nu$ as the intrinsic pulse width, where $\delta\nu$ is the receiver's bandwidth, means that the source's emission is presumed to span a larger range of frequencies than allowed by the observer's filter. This seems to be reasonable in the radio, apart from possibly coherent emission from compact sources like pulsars (Cordes 1976), since our proposed bandwidth is narrow ($d\nu/\nu \sim \text{a few} \times 10^{-5}$; see Equation (21)) and radio

sources tend to have continuum spectra. The same is true in the optical provided one avoids the very narrow, $d\nu/\nu < 20\%$, emission lines, because our recommended $d\nu/\nu$ is $\approx 1/6$ from the words below Equation (35).

After dispersion, this becomes

$$\tilde{g}_x(\omega) = \sqrt{2\pi} a \delta t e^{-[\delta t(\omega - \omega_0)]^2/2} e^{i[k(\omega) - \omega/c]x}. \quad (16)$$

Let us assume that the dispersion effect is only slightly nonlinear, i.e., $k(\omega)$ in the neighborhood of the peak frequency ω_0 is approximately a quadratic function³ of ω :

$$k(\omega) = \frac{\omega}{c} + \frac{1}{2}\beta(\omega - \omega_0)^2. \quad (17)$$

Note that c here does not have to be the speed of light in vacuum. It should represent the velocity of a monochromatic wave of frequency ω_0 in the medium of interest. Thus, \tilde{g} becomes

$$\tilde{g}_x(\omega) = \sqrt{2\pi} a \delta t \exp -\frac{1}{2}[(\delta t)^2 - i\beta x](\omega - \omega_0)^2. \quad (18)$$

Since the effect of dispersion is purely in a phase factor, it does not affect the mean value of the intensity, which is

$$\langle I \rangle = \sqrt{\pi} \lambda \delta t |a|^2. \quad (19)$$

Turning to the intensity correlation function, including both the $\lambda^2(\delta t)^2$ and $\lambda\delta t$ terms of Equations (8) and (10), it is

$$\begin{aligned} \langle I(t)I(0) \rangle - \langle I \rangle^2 &= \lambda^2 |a|^4 \left| \delta t \int d\omega e^{-i\omega t} e^{-[\delta t(\omega - \omega_0)]^2} \right|^2 \\ &+ \frac{\lambda |a|^4}{1 + \xi^2} \int dt' e^{-[(t'+t)^2 + t'^2]/[(\delta t)^2(1+\xi^2)]} \\ &= \langle I \rangle^2 \left\{ e^{-t^2/[2(\delta t)^2]} + \frac{1}{\lambda \delta t \sqrt{2\pi}(1 + \xi^2)} e^{-t^2/[2(\delta t)^2(1+\xi^2)]} \right\}, \end{aligned} \quad (20)$$

where the dispersion stretch factor $\xi = \beta x/(\delta t)^2$ may be written in the context of a uniformly expanding universe as

$$\begin{aligned} \xi &= 8.89 \left(\frac{\delta\nu/\nu}{4.22 \times 10^{-5}} \right)^2 \left(\frac{\nu}{10^9 \text{ Hz}} \right)^{-1} \\ &\times \left(\frac{n_e}{10^{-7} \text{ cm}^{-3}} \right) \left(\frac{\ell}{1 \text{ Gpc}} \right), \end{aligned} \quad (21)$$

with ℓ being the comoving generalization of the propagation distance x , and n_e the mean line-of-sight intergalactic plasma density to the source (see Section 3 of Lieu & Duan 2013, where it is also shown in a table that, away from the Galactic disk directions and neglecting sources with unexpectedly large intrinsic ionized columns, the intergalactic medium dominates the column density $n_e \ell$ to a quasar).

Note that Equation (20) was derived in the context of coherent pulses of radio waves (not photons) by another author, as Equation (12) of Cordes (1976). Moreover, in the $t \ll \delta t \sqrt{1 + \xi^2}$ limit, a simplification was noted by the author in Equation (B6) of Appendix B. When the pulses are as

³ The approximation is correct in the context of plasma dispersion provided $\delta\nu \ll \nu$; see the discussion before Equation (3) of Lieu & Duan (2013).

“microscopic” as photons, however, this limit is actually non-classical because the Heisenberg Uncertainty Principle for finite ξ is $\Delta\nu\Delta t \sim \sqrt{1 + \xi^2}$. We shall henceforth use Equation (20) to predict observable imprints of dispersion by focusing our attention upon intensities measured over timescales $\gg \delta t\sqrt{1 + \xi^2}$ where our hitherto classical treatment of radiation applies. There are two limiting scenarios to discuss. The first is a typical radio passband with $\lambda\delta t \gg 1$, and the second is the visible band (or beyond) with $\lambda\delta t \ll 1$. In each case, the signal is also mixed with noise in different proportion.

4. RADIO OBSERVATIONS

Thus far, we considered a purely signal-limited source detection environment. In radio telescopes, there also exists a system noise that can be converted to an equivalent photon counting rate of μ via the gain of the telescope. Thus, e.g., for Arecibo where the system noise temperature is ~ 35 K and the gain is $G \approx 10$ K Jy $^{-1}$ at $\nu = 1$ GHz (1 Jansky = 10^{-26} W m $^{-2}$ Hz $^{-1}$), this converts to the flux density of 3.5 Jy, or 3.5×10^3 photons s $^{-1}$ Hz $^{-1}$ over the entire telescope’s collecting aperture of 300 m diameter and $\eta = 0.5$ efficiency. The value of the dimensionless quantity $\mu\delta t$ is set by Equation (14) at

$$\mu\delta t = 210 \text{ at } G = 10 \text{ K Jy}^{-1}. \quad (22)$$

Assuming $\mu\delta t \gg 1$ (and $\lambda\delta t \gg 1$) henceforth, and that the local radiation is not dispersed, Equation (20) is modified to

$$\langle I(t)I(0) \rangle - \langle I \rangle^2 = \left[\pi(\lambda + \mu)^2(\delta t)^2 + \sqrt{\frac{\pi}{2}}\mu\delta t \right] |a|^4 e^{-t^2/[2(\delta t)^2]} + \lambda\delta t |a|^4 \sqrt{\frac{\pi}{2(1 + \xi^2)}} e^{-t^2/[2(\delta t)^2(1 + \xi^2)]}. \quad (23)$$

In other words, the unwanted noise is replaced here by a local component of background photons, each being identical in its pulse shape to an intrinsic wave packet from the source.

As indicated in the previous section, the observable imprint of dispersion is upon the variance σ_τ of the average intensity over the macroscopic timescale $\tau \gg \delta t\sqrt{1 + \xi^2}$, which is given by

$$\sigma_\tau^2 = (\delta I_\tau)^2 = \frac{2}{\tau^2} \int_0^\tau dt(\tau - t)[\langle I(t)I(0) \rangle - \bar{I}^2]. \quad (24)$$

It will be shown that σ_τ is modified by dispersion to decreasing degrees as $\tau \rightarrow \infty$.

An important confirmation of the well understood nature of photon fluctuations is obtained by ignoring for the moment the background, i.e., setting $\mu = 0$ in Equation (23), and substituting the result into Equation (24). Enlisting Equation (19), one gets

$$\sigma_\tau^2 = \pi[\sqrt{2\pi}\lambda^2(\delta t)^2 + \lambda\delta t]|a|^4 \frac{\delta t}{\tau} - \sqrt{2\pi}[\sqrt{2\pi}\lambda^2(\delta t)^2 + \lambda\delta t\sqrt{1 + \xi^2}]|a|^4 \left(\frac{\delta t}{\tau}\right)^2. \quad (25)$$

Dropping the higher order $(\delta t)^2/\tau^2$ terms to see if our model reproduces the standard expression for photon noise in vacuum, Equation (25) may be rewritten as

$$\left(\frac{\delta I_\tau}{\bar{I}}\right)^2 = \frac{\bar{n}_\gamma^2 + \bar{n}_\gamma}{\sqrt{2\pi}\lambda^2\delta t\tau}, \quad (26)$$

where the mean photon occupation number per mode is defined at the same value for all modes across the narrow band $\delta\nu \ll \nu$ as

$$\bar{n}_\gamma = \sqrt{2\pi}\lambda\delta t. \quad (27)$$

Now the mean number of photons $\bar{N}_\gamma(t)$ arriving during the interval τ is obviously $\bar{N}_\gamma(\tau) = \lambda\tau$, which must also equal the product of \bar{n}_γ and the number of modes N_{mode} . Equation (27) therefore leads us to the very reasonable result of

$$N_{\text{mode}} = \frac{\tau}{\sqrt{2\pi}\delta t}, \quad (28)$$

indicating that the radiation coherence length is indeed $\approx \delta t$. Moreover, since $(\delta I_\tau/\bar{I})^2 = (\delta N_\gamma/\bar{N}_\gamma)^2 = (\delta N_\gamma)^2/(\lambda^2\tau^2)$, Equations (26) and (28) imply that

$$(\delta N_\gamma)^2 = N_{\text{mode}}(\bar{n}_\gamma^2 + \bar{n}_\gamma). \quad (29)$$

Apart from the small correction terms of the order of $(\delta t/\tau)^2$, then Equation (25) leads to Equation (29), which is in full agreement with Bose–Einstein statistics.

This demonstrates the validity of the classical treatment presented. In fact, it has been shown that, for ordinary (chaotic) light from a steady source, the classical and full quantum calculations of statistical correlations give the same results (see, e.g., page 1, Chapter 5 of Loudon 2000; also Equation (103) and the remarks thereafter of Baltz 2003).

In the “radio” limit of $\bar{n}_\gamma \approx \lambda\delta t \gg 1$,

$$\left(\frac{\delta I_\tau}{\bar{I}}\right)^2 = \left(\frac{\delta N_\gamma}{\bar{N}_\gamma}\right)^2 \approx \frac{1}{N_{\text{mode}}} = \frac{1}{2\sqrt{\pi}\tau\delta\nu}, \quad (30)$$

or $(\delta I_\tau/\bar{I})^2 \approx 1/(\tau\delta\nu)$, in accordance with the radiometer equation.

The method of detection utilizes the behavior of σ_τ^2 at $\tau \gg \delta t\sqrt{1 + \xi^2}$, when Equation (24) can be evaluated with the aid of Equation (23) to become

$$\sigma_\tau^2 = \pi[\sqrt{2\pi}(\mu + \lambda)^2(\delta t)^2 + (\mu + \lambda)\delta t]|a|^4 \frac{\delta t}{\tau} - [2\pi(\mu + \lambda)^2(\delta t)^2 + \sqrt{2\pi}(\mu + \lambda\sqrt{1 + \xi^2})\delta t]|a|^4 \left(\frac{\delta t}{\tau}\right)^2. \quad (31)$$

The ratio of the dispersive-dependent correction term that suppresses the variance (the preceding $\mu\delta t|a|^4$ term is usually negligible wrt this term and can, in any case, be subtracted) to the main term is

$$r = -\frac{\xi\lambda}{\pi(\mu + \lambda)^2\tau} = -2.62 \times 10^{-5} \left(\frac{T_s}{10 \text{ K}}\right) \times \left(\frac{T_s + T_{\text{sys}}}{45 \text{ K}}\right)^{-2} \left(\frac{\xi\delta t/\tau}{0.1}\right), \quad (32)$$

where the default value of τ is set at $\tau = 10\xi\delta t$ with $\xi \gg 1$ as given in Equation (21), and $T_s = 10$ K is because we assumed the default strength of the source to be 1 Jy, i.e., $\lambda = \mu$ and both are given by Equation (22).

We turn to signal-to-noise issues. Apart from the variance of the intensity on the timescale τ , the other ingredient to a successful measurement here is an accurate determination

of the mean intensity,⁴ so that all the ξ -independent terms of Equation (31) are known and can be subtracted from the variance to find the residual. For this purpose, one can assume that in the $\lambda\delta t \gg 1$ and $\mu\delta t \gg 1$ limit, the intensity varies according to the leading term of Equation (31) as $\sigma_\tau \approx \langle I \rangle \delta t / \tau$, which is $\ll \langle I \rangle$ for $\tau \gg \delta t$. Thus, the fluctuations may be regarded as normal (Gaussian). From Equations (A5) and (A6), the variance is the main source of error and is accurate to the fractional uncertainty of $\delta\sigma_\tau^2 / \sigma_\tau^2 = \sqrt{2/N_s}$, or 4.47×10^{-6} if $N_s = 10^{11}$. Such a value of N_s may be obtained by repeated sampling to a total exposure time of $t_{\text{exp}} = 10^4$ s, or three hours, at intervals of $\tau = 10\xi\delta t \approx 0.237$ ms as suggested in Equation (32), also using the channel width of $\delta\nu = 42.2$ kHz in Equation (32) to simultaneously cover a total bandwidth $\Delta\nu = 0.1$ GHz at the central frequency $\nu = 1$ GHz. In precise terms, $N_s = t_{\text{exp}}\Delta\nu / (\tau\delta\nu) = t_{\text{exp}}\Delta\nu / (10\xi\delta t\delta\nu) = 10^{11}$ under this scenario. Note also that since the denominator $10\xi\delta t\delta\nu \sim (\delta\nu)^2$, we have $N_s \sim 1/(\delta\nu)^2$ for fixed t_{exp} and $\Delta\nu$, hence the quality of a detection is favored by using a narrower channel $\delta\nu$ whilst maintaining the $\xi \gg 1$ criterion of significant dispersion. Comparing this uncertainty of $\sqrt{2/N_s}$ with the signal strength of Equation (32), one sees that r can be detected at the significance of $\approx 5.91\sigma$, increasing to 8.55σ at the optimal source brightness of 3.5 Jy where the product of the two temperature dependent coefficients in Equation (32) is at its maximum.

In fact, using Equation (32), the signal-to-noise ratio may be written as

$$\begin{aligned} \frac{|r|}{\delta r} &= 5.91 \left(\frac{T_s}{10 \text{ K}} \right) \left(\frac{T_s + T_{\text{sys}}}{45 \text{ K}} \right)^{-2} \left(\frac{\xi\delta t/\tau}{0.1} \right)^{3/2} \left(\frac{\delta\nu}{160 \text{ kHz}} \right)^{-1} \\ &\times \left(\frac{\Delta\nu}{0.1 \text{ GHz}} \right)^{1/2} \left(\frac{\nu}{1 \text{ GHz}} \right)^{3/2} \left(\frac{t_{\text{exp}}}{10^4 \text{ s}} \right)^{1/2} \\ &\times \left(\frac{n_e \ell}{10^{-7} \text{ cm}^{-3} \text{ Gpc}} \right)^{-1/2}. \end{aligned} \quad (33)$$

One caveat to be mindful of is that the weakness of the signal r means that the actual detection significance will be reduced by extra (system) noise, including radio frequency interference and especially gain variations, which have the effect of increasing the background μ typically by 10%; see Tuccari (2009).

We should also discuss the potential complication of scattering by plasma clumps, which can also broaden radio wave packets in all three dimensions. However, for a mean intergalactic plasma column density appropriate to a 1 Gpc source, a wave packet is typically broadened to last the duration of $\approx 10^{-6}$ s, (Bhat et al. 2004). This is still very small compared with the $\approx 10^{-4}$ s of dispersion effect in the propagation direction x , viz. (21). We shall therefore neglect it.

Given the formidability of securing a robust detection in the radio, it may be advisable to first perform a feasibility test by observing an extragalactic source at low Galactic latitude b where dispersion by the interstellar medium (ISM) sets a ‘‘bottom line’’ minimum in the plasma column $n_e \ell$ (hence ξ) of the order of

$$n_e^{\text{ISM}} \ell > 10^{-7} \text{ cm}^{-3} \text{ Gpc}, \quad \text{for } \ell \gtrsim 1.5 \text{ kpc}, \quad (34)$$

⁴ It is also necessary to know the occupation number of the source and background, i.e., $\mu\delta t$ and $\lambda\delta t$, which constitute the second term on the right side of Equation (31), but this can be determined to the same accuracy as $\langle I \rangle$ provided the passband filter defining $\delta\nu$ is stable (which should not be an issue because neither the spectra of quasars nor the background has sharp features to enable any drift in the passband to cause a correspondingly large change in the occupation numbers).

(see, e.g., Table 1 of Davidson & Terzian 1969), which, according to Table 1 below, is comparable to the IGM column, thereby affording one a rough idea of what result to expect. We suggest looking at a low b quasar such as J2109+353, which has $(l, b) = (80.3, -8.35)$ and a flux of 1.2 Jy at 1.4 GHz; see Table 1 of Im et al. (2007). The reason for avoiding any Galactic sources is that these will inevitably have to be pulsars, except their emission may occur in a highly unusual form as coherent bunches of photons (Cordes 1976) and therefore the receiver’s passband $1/\delta\nu$ may underestimate the intrinsic wave packet size and jeopardize the measurement; see Section 6 for further discussions.

The conclusion is that even with the largest radio telescope available, observations in this band can at best yield a marginal detection of cosmological dispersion with a set of fortuitous and very fine tuned parameters. It is hard to see how to develop the technique to become a utility for mapping the baryonic content of the entire near universe. To achieve that, one must explore other wavelengths.

5. OPTICAL OBSERVATIONS

In the opposite limit of $\lambda\delta t \ll 1$ and $\mu\delta t \ll 1$, such as optical observation of quasars, Equation (23) simplifies to

$$\begin{aligned} \langle I(t)I(0) \rangle - \langle I \rangle^2 &= \sqrt{\frac{\pi}{2}} \mu \delta t |a|^4 e^{-t^2/[2(\delta t)^2]} \\ &+ \lambda \delta t |a|^4 \sqrt{\frac{\pi}{2(1+\xi^2)}} e^{-t^2/[2(\delta t)^2(1+\xi^2)]}. \end{aligned} \quad (35)$$

For typical observation through a Gaussian V-band filter of $\nu = 500$ THz and $\delta\nu/\nu = 1/6$ (or 0.1 μm passband at 0.6 μm wavelength) one has, by Equations (14) and (21),

$$\delta t \sqrt{1+\xi^2} \approx 3.75 \times 10^{-13} \text{ s}, \quad \text{at } \xi = 278 \quad (36)$$

for a source ≈ 1 Gpc away.

The imprint of dispersion upon the autocorrelation function is not affected at optical wavelengths by the (undesirable) $\lambda^2(\delta t)^2$ and $\mu^2(\delta t)^2$ terms, as such terms are negligible in the Poisson limit of photon noise. There is also the advantage of the generally higher signal-to-background ratio λ/μ here (see below). However, the low photon count rate, the narrowness of the visible band, and the smaller value of ξ present this type of observations with its own challenges.

Like the radio band, one may proceed to measure the dispersive column density via the variance of intensity fluctuations on timescales longer than $\delta t \sqrt{1+\xi^2}$, by applying Equation (35) to Equation (24) to get

$$\begin{aligned} \sigma_\tau^2 &= (\delta I_\tau)^2 = \pi(\mu + \lambda)\delta t |a|^4 \frac{\delta t}{\tau} - \sqrt{2\pi}(\mu + \lambda\sqrt{1+\xi^2})\delta t |a|^4 \\ &\times \left(\frac{\delta t}{\tau} \right)^2, \quad \text{for } \tau \gg \delta t \sqrt{1+\xi^2}. \end{aligned} \quad (37)$$

As already discussed in the last section, the $O(\delta t/\tau)$ term spells Poisson statistics in this limit. To check this again, note that when Equation (37) is used in conjunction with Equation (19) this lowest order contribution to the variance satisfies the relation

$$\left(\frac{\delta I_\tau}{\bar{I}} \right)^2 = \left(\frac{\delta N_\nu}{\bar{N}_\nu} \right)^2 = \frac{1}{(\lambda + \mu)\tau} = \frac{1}{\bar{N}_\nu}, \quad (38)$$

Table 1

Observation of Dispersion Induced “Photon Anti-bunching” to Determine the Line-of-sight Plasma Column Density to Astrophysical Sources in the V Band

mV of Source	Obs.	λ (s^{-1})	Column $n_e \ell$ (cm^{-3} Gpc)	τ (ns)	Statistical Significance	$\delta n_2/n_2$
10 (star)	Lick 3m	4×10^6	10^{-8}	1	47.2σ	$6 \times 10^5/8 \times 10^7$
13 (3C273)	Lick 3m	2.5×10^4	10^{-7}	10	13.5σ	3, 750/38, 642
15 (quasar)	Keck 10m	4×10^4	10^{-7}	10	14.0σ	6, 000/91, 592
18 (quasar)	Keck 10m	2,560	3×10^{-7}	20	7.59σ	576/2, 873

Notes. In each case, the night sky background is assumed to be 2800 photons s^{-1} , and the exposure time is 10^4 s. The last column gives the δn_2 reduction in the number of double photon time bins (of width τ) due to dispersion, and the expected number n_2 of such bins in the limit of no dispersion. The statistical significance is computed with the use of a beam splitter in mind to counter dead time problem (see Section 5.2), i.e., the number of σ is smaller than $\delta n_2/\sqrt{n_2}$ by a factor of $\sqrt{2}$ as a result.

independent of dispersion, where N_γ is the total number of arriving photons in the interval $\tau \gg \delta t \sqrt{1 + \xi^2}$. This result is a direct consequence of Equations (28) and (29) in the “optical” limit of $\bar{n}_\gamma \approx (\mu + \lambda)\delta t \ll 1$ (end of Section 2).

Over this same interval τ , however, the imprint of dispersion is upon the modification to Poisson ($\sigma_\tau^2 \propto 1/\tau$) fluctuations from the very last term of Equation (37). Explicitly, the ratio of this term to the rest of σ_τ^2 is, assuming $\xi \gg 1$,

$$r = -\sqrt{\frac{2}{\pi}} \frac{\xi \lambda}{\mu + \lambda} \frac{\delta t}{\tau} = -3 \times 10^{-5} \left(\frac{\tau}{10 \text{ ns}}\right)^{-1} \times \left(\frac{\delta \nu}{83.3 \text{ THz}}\right) \left(\frac{\nu}{500 \text{ THz}}\right)^{-3} \left(\frac{n_e \ell}{10^{-7} \text{ cm}^{-3} \text{ Gpc}}\right), \quad (39)$$

which is ξ -dependent⁵ (the last expression is written with $\mu = 0$ in mind). Thus, one expects to see a significantly larger non-Poisson correction to the curve when there is dispersion, signal-to-noise-permitting, in the form of a suppression of the variance $(\delta N_\gamma)^2$ of the photon counts from the Poisson value of $(\delta N_\gamma)^2 = \bar{N}_\gamma$.

5.1. Photon Counting on Short Timescales

Our calculation is done in the context of the 10 m telescope at the Keck Observatory, although the formulae are presented in such a way as to facilitate adaptation to another environment, and the final results as shown in Table 1 contain sources observed by the Lick 3 m as well. Assuming a quantum efficiency of 16% (Hamamatsu Photonics K.K. Editorial Committee 2006) for the photomultiplier detector, a quasar with V-band (see the beginning of this section for filter specifics) magnitude mV = 15 would deliver $\lambda = 4 \times 10^4$ photons s^{-1} , against a background of $\mu = 2,800$ photons s^{-1} if we assume the dark sky background has the surface brightness of 20 mV arcsec^{-2} and a point spread function of 3'' diameter for the quasar. Thus the data are signal-dominated, with $\lambda \gg \mu$.

We explore the prospect of using the above parameters for an observing run. Suppose one of them is exposed⁶ for a time $t_{\text{exp}} = 10^4$ s (≈ 3 hr), with the arrival time of individual photons recorded at the time resolution of $\tau = 10$ ns (10^{-8} s, photomultiplier tubes (PMTs) with this requisite response time and without dead time issues are available; see below), i.e., there

⁵ Note that when calculating r , we ignored the $\mu(\delta t)^3/\tau^2$ term in Equation (37) since it is much less than the $\mu(\delta t)^2/\tau$ term there.

⁶ The exposure time t_{exp} must satisfy the condition $\lambda t_{\text{exp}} \gg 1$; see Section 1. The inequality ensures that there are many source photons detected during the observation.

are $N_s = t_{\text{exp}}/\tau \approx 10^{12}$ time bins to measure the variance σ_τ^2 . Neglecting μ in Equation (39), the imprint of dispersion is an excess of the mean photon count per bin from the variance by the fractional amount $r = -3 \times 10^{-5}$. Now, from Equation (A12) of the Appendix, the random error in this fraction is

$$\delta r \approx \sqrt{\frac{2}{N_s}} = 1.4 \times 10^{-6} \left(\frac{\tau}{10 \text{ ns}}\right)^{1/2} \left(\frac{t_{\text{exp}}}{10^4 \text{ s}}\right)^{-1/2}. \quad (40)$$

The signal-to-noise ratio is obtained by combining Equation (39) with Equation (40), as

$$\frac{|r|}{\delta r} = 19.8 \left(\frac{\lambda}{4 \times 10^4 \text{ s}^{-1}}\right) \left(\frac{\lambda + \mu}{4.28 \times 10^4 \text{ s}^{-1}}\right)^{-1} \left(\frac{\tau}{10 \text{ ns}}\right)^{-3/2} \times \left(\frac{t_{\text{exp}}}{10^4 \text{ s}}\right)^{1/2} \left(\frac{\delta \nu}{83.3 \text{ THz}}\right) \left(\frac{\nu}{500 \text{ THz}}\right)^{-3} \times \left(\frac{n_e \ell}{10^{-7} \text{ cm}^{-3} \text{ Gpc}}\right). \quad (41)$$

Hence, for the prescribed observing conditions, one can expect a $\approx 20\sigma$ detection of the dispersion signal, and a tight ensuing constraint on the intervening IGM column density to the quasar via ξ .

The reader can also verify readily that under the above scenario the $\bar{n}_2 \approx 9.16 \times 10^4$ “double photon” bins expected from a pure Poisson distribution is sufficiently large to enable a slight reduction in the number n_2 of such bins to produce the corresponding suppression of the variance to a sub-Poisson value beneath the mean. To be precise, the variance of N_γ will come down from the mean by the fraction of 3×10^{-5} if the average number of “double photon” bins is $\bar{n}_2 = 8.56 \times 10^4$ instead of $\bar{n}_2 = 9.16 \times 10^4$. In another way of understanding why this decrease is 20σ significant, one could take the difference between the above two values of n_2 and divide it by the statistical fluctuation of n_2 , viz. $\sqrt{\bar{n}_2} \approx 303$. More generally, it can be shown without too much difficulty that for $\tau \gg \delta t \sqrt{1 + \xi^2}$,

$$\bar{n}_2 = \frac{1}{2}(\lambda + \mu)^2 \tau^2 N_s - \sqrt{\frac{1 + \xi^2}{2\pi}} \lambda \delta t N_s, \quad (42)$$

from which one can see that the slight sub-Poisson behavior due to the last term is enhanced by dispersion.

A sensitivity limit of this technique exists, however, when there are too few arriving photons from a faint source, i.e., if the mean count per bin $\bar{m} = \lambda \tau$ is so low that the first term on the right side of Equation (42) becomes as small as the second.

Since n_2 obviously cannot be negative, the dispersion signal is compromised once this limit is reached, and increasingly more so beyond it. Explicitly, the smallest variance afforded by any photon counting data of a given mean \bar{m} occurs when $\bar{n}_2 = 0$ and $\bar{n}_1 = \bar{m} - \bar{m}^2$, and the variance of the configuration lies below the mean by the fractional amount $(\bar{m} - \sigma^2)/\bar{m} = \bar{m}$. If $\bar{m} < |r|$ (where r is given by Equation (39)), the measurement will not be able to recover the full signal. Thus, a faint source will require large τ to satisfy the $\bar{n}_2 \gg 1$ criterion for full signal detection, but Equation (41) asserts that as τ increases the signal-to-noise ratio deteriorates. The use of too large a time window τ would also put one closer to the timescales of atmospheric turbulence effects, which tends to induce super-Poisson variance at the few percent level over durations of 1 ms to 1 s (Tokovinin et al. 2003), i.e., although this concern is not imminent it should be kept in mind. In general, then, a correspondingly longer exposure time, $t_{\text{exp}} = N_s \tau$, is needed to maintain the signal-to-noise ratio $|r|/\delta r$. Since, for \bar{n}_2 of Equation (42) to stay positive, τ must at least be sufficiently large to enable $\lambda \tau$ to match the constant $\approx \sqrt{\xi} \lambda \delta t$, viz.

$$\tau \gg 7.75 \times 10^{-10} \left(\frac{\lambda}{4 \times 10^4 \text{ s}^{-1}} \right)^{-1/2} \left(\frac{\delta \nu}{83.3 \text{ THz}} \right)^{1/2} \times \left(\frac{\nu}{500 \text{ THz}} \right)^{-3/2} \left(\frac{n_e \ell}{10^{-7} \text{ cm}^{-3} \text{ Gpc}} \right)^{1/2} \text{ s}, \quad (43)$$

one sees from Equation (41) that in order to maintain $|r|/\delta r$ the exposure time, t_{exp} , must be raised for a faint source according to the scaling⁷ $t_{\text{exp}} \sim 1/\lambda^{3/2}$ (alternatively, the telescope aperture must enlarge by the same scaling), because in Equation (41) $t_{\text{exp}} \sim \tau^3$ for a given $|r|/\delta r$ and $\tau_{\text{min}} \sim \lambda^{-1/2}$ from Equation (43). Moreover, eventually the background μ will also become significant, in which case t_{exp} will have to be even larger, although the fact that faintness usually implies remoteness would work in one’s favor because it means $|r|$ is larger as a result of the column ℓ being so.

Although the visible regime appears more promising than the radio, it is still advisable to test the effect, which remains rather feeble, using a “calibration” source as described in the end of Section 4, where we also explained why a source located 1.5 kpc or more away in the direction of the Galactic disk would suffice. Specifically, one can enlist here a bright disk star (i.e., pulsars are not the only choice even if one observes only sources within the Galaxy), such as P Cygni (34 Cyg) which is 1.8 kpc away (Balan et al. 2010) but has $mV = 4.8$ from the SIMBAD Astronomical Database. Signal-to-noise estimates for a fainter ($mV = 10$) star are given in Table 1, from which it can be seen that P Cygni should be a relatively easy target.

5.2. An Outline of the Basic Design

Here we present the elements of a feasible observational scheme. Our intention here is not to define in detail the optimal technique, however. This is work in progress, which will appear in a separate publication.

Our goal is to measure the reduction of n_2 , which can be done by repeatedly counting the number of photons received by the detector using a fixed counting cycle (or window span). After a sufficient number of photons have been received, a histogram of the per-cycle photon count can be developed and is

⁷ This scaling is to be contrasted with then exposure $t_{\text{exp}} \sim 1/\lambda$ necessary to detect a source photometrically to a certain level of significance under the same signal-limited condition as here.

compared to the Poisson distribution. The temporal width of the counting cycle has to be chosen in such a way that the expected number of “double photon” cycles is statistically robust while the random noise is kept low to ensure an appropriate signal-to-noise ratio, as already explained. Note that the latter requirement usually implies that the “triple photon” probability has to be prohibitively low, which leaves n_2 as the only parameter that carries the signature of the sub-Poisson distribution.

Several practical issues associated with photon counters should be taken into account, including dead time, afterpulsing, and dark counts. The dead time of a photon counter tends to make a perfect Poisson distribution appear to be sub-Poisson. Such an effect could mask the sub-Poisson distribution due to dispersion if proper care is not taken. For example, the counting circuits of a PMT usually introduce a dead time of a few tens of nanoseconds. Such a photon counter would not be able to capture any “double photon” counts if a 1–10 ns counting cycle is used. To address this problem, we propose to use two identical PMT counters separated by a 50:50 beamsplitter. The outputs of the two counters are combined with a sub-ns timing error to form one single stream of photon counts before being gated by a clock. To simplify the data analysis, we can further make the counting cycle same as the counter dead time (by adjusting the PMT electronics and the clock frequency). Such a detection system in principle can capture all the “single photon” counts and half of the “double photon” counts. The latter is because 50% of the photon pairs arriving within a counting cycle fall onto different PMTs and therefore can effectively be tallied with a worsening in the signal-to-noise by the factor of $\sqrt{2}$.

On the other hand, afterpulsing and dark counts tend to make a perfect Poisson distribution appear to be super-Poisson by adding artificial multi-photon counts. However, with the help of an ideal Poisson source, both effects can be fairly well calibrated and subsequently taken out of the dispersion measurement. The recent development of hybrid photodetectors also offers a detector with almost zero afterpulsing (Suyama 2008). The counter calibration can be done off-line in the lab. Alternatively, the task can also be done on site by pointing the telescope first to a nearby star and attenuating the incoming photon intensity to a level comparable to the intended quasar.

In Table 1, we show the typical plasma column density $n_e \ell$ to various astrophysical sources, the first of which is non-cosmological (a nearby star), and the recommended parameters at the Lick or Keck Observatory (the former restricted to stars and the brightest quasar 3C273 as test beds) to measure $n_e \ell$ via ξ .

6. CONCLUSION

In respect of the quest for the missing baryons in the near universe, an effort at the forefront of contemporary cosmology, we proposed and developed a new technique that complements the current reliance on fast radio transients. By examining the detailed statistical fluctuations of the electromagnetic radiation arriving from distant quasars, we showed that it is possible to detect the imprint of dispersion and infer the total line-of-sight column density of partially ionized (i.e., the bulk of the missing) baryons to the source.

The technique as applied to the radio and visible bands is discussed. For each case it is found that different challenges exist. In the radio, systematic noises like the broad band interference must be kept to a minimum, and random errors reduced by adequate exposure using simultaneously many

frequency channels. In the visible, the background is less of a problem, rather the scarcity of photons and the relatively smaller effect of dispersion. However, all these issues can potentially be overcome, and the prospect of success is reachable. Overall, the visible band appears more promising, and could be developed into a widely applicable utility for mapping the low z plasma columns across the sky.

The authors acknowledge Eric Korpela, Michael Lampton, and Stuart Bowyer for discussions.

APPENDIX

SAMPLE VARIANCE DISTRIBUTION: CORRELATION BETWEEN SAMPLE MEAN AND VARIANCE

For N_s samples (each is in the context of this paper a measurement of either radio intensity or optical photon counts over some small interval of time) of a variate n drawn from a parent population of central moments μ_m , $m = 1, 2, \dots$, viz.

$$\mu_j = \langle (m - \mu)^j \rangle = \frac{1}{N_s} \sum_{i=1}^{N_s} (n_i - m)^j, \quad (\text{A1})$$

where $\mu = \langle m \rangle$ is the true mean (not to be confused with the background count rate μ of Section 4), and the sample mean and variance are denoted by m and σ^2 , respectively, and are defined as

$$m = \frac{1}{N_s} \sum_{i=1}^{N_s} n_i; \quad \sigma^2 = \left(\frac{1}{N_s} \sum_{i=1}^{N_s} n_i^2 \right) - m^2. \quad (\text{A2})$$

The expectation value for the sample variance, $\langle \sigma^2 \rangle = (N_s - 1)\mu_2/N_s$ is $\approx \mu_2$ to a fractional error $\approx 1/N_s$, and that of the variance of the sample variance is

$$\langle (\delta\sigma^2)^2 \rangle = \frac{(N_s - 1)^2}{N_s^3} \mu_4 - \frac{(N_s - 1)(N_s - 3)}{N_s^3} \mu_2^2, \quad (\text{A3})$$

which also simplifies to

$$\langle (\delta\sigma^2)^2 \rangle = \frac{\mu_4 - \mu_2^2}{N_s}, \quad (\text{A4})$$

again to correct a fractional error $\approx 1/N_s$.

Let us now apply Equation (A4). For a normal distribution where $\mu_2 = \sigma^2$ and $\mu_4 = 3\sigma^4$, it implies, adopting slightly loose notations,

$$\frac{\delta\sigma^2}{\langle \sigma^2 \rangle} = \sqrt{\frac{2}{N_s}}. \quad (\text{A5})$$

This is to be contrasted with the standard expression for the uncertainty in the mean,

$$\frac{\delta m}{\langle m \rangle} = \frac{\sigma}{\sqrt{N_s}}, \quad (\text{A6})$$

which is much smaller in the limit $\sigma \ll \mu$.

For Poisson fluctuations, the variance $\mu_2 = \sigma^2 = \mu$ and $\mu_4 = \mu(1 + 3\mu)$, and it readily follows from Equation (A4) that the relative uncertainty $\delta\sigma^2/\sigma^2$ in the sample variance σ^2 is

$$\frac{\delta\sigma^2}{\langle \sigma^2 \rangle} = \frac{1}{\sqrt{N_s\mu}}, \quad (\text{A7})$$

and correct to the relative accuracy $\approx 1/N_s$. Equation (A7) is the same as the relative uncertainty in the sample mean, $\delta m/\langle m \rangle$, and this peculiar feature is specific to the $\mu \ll 1$ limit only. In the opposite limit, the sample variance fluctuates much more than the mean.

For completeness, even though the next result is not used in the paper, we include phase-noise fluctuations of radio observations, i.e., the $\mu \gg 1$ limit of Bose–Einstein statistics where $\mu_2 = \mu^2$ and $\mu_4 = \mu(1 + \mu)(1 + 9\mu + 9\mu^2) \approx 9\mu^4$, one instead obtains the relation

$$\frac{\delta\sigma^2}{\langle \sigma^2 \rangle} = \sqrt{\frac{8}{N_s}} \quad (\text{A8})$$

for the relative uncertainty in $\langle \sigma^2 \rangle$ from Equation (A4).

We are also interested in the relative uncertainty in the difference between the sample mean and variance, viz. $\delta(m - \sigma^2)/\sigma^2$ of a Poisson distribution. To begin, let us express the variance of $x = m - \sigma^2$ as

$$\langle (\delta x)^2 \rangle = \langle (\delta m)^2 \rangle + \langle (\delta\sigma^2)^2 \rangle - 2 \text{cov}(m, \sigma^2), \quad (\text{A9})$$

where the covariance function of two variates y and z is defined in the usual manner as

$$\text{cov}(y, z) = \langle yz \rangle - \langle y \rangle \langle z \rangle.$$

In this case, since the Poisson $\langle m \rangle = \langle \sigma^2 \rangle = \mu$, we may write

$$\text{cov}(m, \sigma^2) = \langle m\sigma^2 \rangle - \mu^2.$$

This equation may be recast in terms of the new variates $\tilde{n}_i = n_i - \mu$ and $\tilde{m} = m - \mu$, as

$$\text{cov}(m, \sigma^2) = \langle \tilde{m}\sigma^2 \rangle = \langle \tilde{n}_1\sigma^2 \rangle = \frac{1}{N_s} \langle \tilde{n}_1 w \rangle, \quad (\text{A10})$$

where

$$w = \sum_{i=1}^{N_s} (\tilde{n}_i - \tilde{m})^2.$$

Now w is a sum of the products \tilde{n}_i^2 and $\tilde{n}_i\tilde{n}_j$ for $i \neq j$. Most of these products actually do not affect the calculation of $\langle \tilde{n}_1 w \rangle$, since, e.g., $\langle \tilde{n}_1\tilde{n}_i\tilde{n}_j \rangle = 0$ for $i \neq j$ and $\langle \tilde{n}_1\tilde{n}_i^2 \rangle = 0$ for $i \neq 1$.

The quantity $\langle \tilde{n}_1 w \rangle$ may therefore be simplified to read $\langle \tilde{n}_1 w \rangle = \alpha \langle \tilde{n}_1^3 \rangle$, where $\alpha = 1$ is accurate to the relative error of $1/N_s$ and $\langle \tilde{n}_1^3 \rangle = \mu_3 = \mu$ is the third moment of the Poisson distribution. Substituting into Equation (A10), we obtain

$$\text{cov}(m, \sigma^2) = \frac{\mu}{N_s}. \quad (\text{A11})$$

Since $\langle (\delta m)^2 \rangle = \langle \sigma^2 \rangle/N_s = \mu/N_s$ and $\langle (\delta\sigma^2)^2 \rangle = (\mu + 2\mu^2)/N_s$ from Equation (A4) and the fact that the Poisson $\mu_4 = \mu + 3\mu^2$ and $\mu_2 = \mu$, these results may be applied along with Equation (A11) to Equation (A9) to arrive at

$$\langle (\delta x)^2 \rangle = \frac{2\mu^2}{N_s}. \quad (\text{A12})$$

In the limit of small counts per bin, i.e., $n \approx \mu \ll 1$, the variance $\langle (\delta x)^2 \rangle$ in the difference between the sample mean and variance of a Poisson distribution, $n - \sigma^2$ is much smaller than the μ/N_s variance in the sample mean and variance themselves.

REFERENCES

- Ade, P. A. R., Aghanim, N., Armitage-Caplan, C., et al. 2013, *A&A*, submitted (arXiv:1303.5076)
- Afshordi, N., Lin, Y.-T., Nagai, D., & Sanderson, A. J. R. 2007, *MNRAS*, **378**, 293
- Arnaud, M., Pointecouteau, E., & Pratt, G. W. 2007, *A&A*, **474**, L37
- Balan, A., Tycner, C., Zavala, R. T., et al. 2010, *AJ*, **139**, 2269
- Baltz, R. V. 2003, 20th Course of the International School of Atomic and Molecular Spectroscopy, ed. B. Di Bartolo & O. Forte (Dordrecht: Kluwer)
- Becker, R. H., Fan, X., White, R. L., et al. 2001, *AJ*, **122**, 2850
- Bhat, N. D., Cordes, J. M., Camilo, F., Nice, D. J., & Lorimer, D. R. 2004, *ApJ*, **605**, 759
- Burles, S., & Tytler, D. 1998, *SSRv*, **84**, 65
- Bonamente, M., Lieu, R., Joy, M. K., & Nevalainen, J. H. 2002, *ApJ*, **576**, 688
- Born, M., & Wolf, E. 1970, *Principles of Optics* (4th ed.; Oxford: Pergamon)
- Cen, R., & Ostriker, J. P. 1999, *ApJ*, **514**, 1
- Cordes, J. M. 1976, *ApJ*, **210**, 780
- Davé, R., Cen, R., Ostriker, J. P., et al. 2001, *ApJ*, **552**, 473
- Davidson, K., & Terzian, Y. 1969, *ApJ*, **74**, 849
- Einstein, A. 1917, *PhysZ*, **18**, 121
- Elvis, M. 2002, in *ASP Conf. Proc. 258, Issues in Unification of AGN*, ed. R. Maiolino, A. Marconi, & N. Nagar (San Francisco, CA: ASP), 51
- Ettori, S., Morandi, A., Tozzi, P., et al. 2009, *A&A*, **501**, 61
- Fukugita, M., Hogan, C. J., & Peebles, P. J. E. 1998, *ApJ*, **503**, 518
- Hamamatsu Photonics K.K. Editorial Committee 2006, *Photomultiplier Tubes—Basics and Applications* (Hamamatsu Photonics K.K. Electron Tube Division), 32
- Hanbury Brown, R., & Twiss, R. Q. 1957, *RSPSA*, **242**, 300
- Im, M., Lee, I., Cho, Y., et al. 2007, *ApJ*, **664**, 64
- Komatsu, E., Smith, K. M., Dunkley, J., et al. 2011, *ApJS*, **192**, 18
- Landry, D., Bonamente, M., Giles, P., Maughan, B., & Joy, M. 2013, *MNRAS*, **433**, 2790
- Lieu, R., & Duan, L. 2013, *ApJL*, **763**, L44
- Lorimer, D. R., Bailes, M., McLaughlin, M. A., Narkevic, D. J., & Crawford, F. 2007, *Sci*, **318**, 777
- Lorimer, D. R., Karastergiou, A., McLaughlin, M. A., & Johnston, S. 2013, *MNRAS*, in press (arXiv:1307.1200)
- Loudon, R. 2000, *Quantum Theory of Light* (3rd ed.; Oxford: Oxford Univ. Press)
- Mandel, L. 1958, *PPS*, **72**, 1037
- McKernan, B., Yaqoob, T., & Reynolds, C. S. 2007, *MNRAS*, **379**, 1359
- McQuinn, M. 2013, *ApJ*, submitted (arXiv:1309.4451)
- Mittaz, J. P. D., Lieu, R., & Lockman, F. J. 1998, *ApJL*, **498**, L17
- Press, W. H., & Schechter, P. 1974, *ApJ*, **187**, 425
- Purcell, E. M. 1956, *Natur*, **178**, 1449
- Rauch, M. 1998, *ARAA*, **36**, 267
- Simionescu, A., Allen, S. W., Mantz, A., et al. 2011, *Sci*, **331**, 1576
- Sun, M., Voit, G. M., Donahue, M., et al. 2009, *ApJ*, **693**, 1142
- Suyama, M., & Lares, M. 2008, *Laser Focus World*, **44**, 80
- Thornton, D., Stappers, B., Bailes, M., et al. 2013, *Sci*, **341**, 53
- Tokovinin, A., Kornilov, V., Shatsky, N., & Voziakova, O. 2003, *MNRAS*, **343**, 891
- Tuccari, G. 2009, *Interference at a VLBI Station: Analysis and Mitigation* (Istituto di Radioastronomia-INAFA)
- Vikhlinin, A., Kravtsov, A., Forman, W., et al. 2006, *ApJ*, **640**, 691
- White, S. D. M., & Rees, M. J. 1974, *ApJ*, **187**, 449
- Zavala, J., Vogelsberger, M., & Walker, M. G. 2013, *MNRAS*, **431**, 20