Turbulence Transport Model Applied
Space Physics And Astrophysics

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Background

-Magnetohydrodynamics (MHD) Turbulence: Magneto – magnetic fluid hydro - Liquid dynamics - movement

-MHD deals with dynamics of electrically conducting fluids.

-MHD Turbulence is observed when the Reynolds number of magnetofluid is large.
Background

- Sun - source of solar wind which is highly inhomogeneous magnetofluid expanding radially outward from it.
- The velocity and magnetic field in terms of fluctuating fields,

\[ V = \langle V \rangle + \nu \]

\[ B = \langle B \rangle + b \]

Where \( \nu \) is fluctuation on mean velocity \( \langle V \rangle \) and \( b \) is fluctuation on mean magnetic field \( \langle B \rangle \).
- Elsasser variables (Elsasser 1950),

\[ z^\pm = u \pm \frac{b}{\sqrt{4\pi \rho}} \]

Where \( z^+ \) and \( z^- \) describe wave propagating outward and inward with respect to sun.
- The transport equation of evolution of fluctuations of \( u \) and \( b \) from their mean velocity and magnetic field in terms of elsasser variables (Zhou and Matthaeus, 1990a,b)

\[ \frac{\partial z^\pm}{\partial t} + (U \mp V_A) \cdot \nabla z^\pm + \frac{1}{2} \nabla \cdot (U/2 \pm V_A) z^\pm + z^\mp \cdot \left[ \nabla U \pm \frac{\nabla B}{\sqrt{4\pi \rho}} - \frac{1}{2} I \nabla \cdot (U/2 \pm V_A) \right] = NL_{\pm} + S^\pm \]

Where \( I \) is the identity matrix, \( NL_{\pm} \) are the non-linear terms, \( S^\pm \) is the source term.
- Transport equations of Normalized energy density of magnetic fluctuation \( \bar{E}_b \) and correlation length \( \bar{\lambda} \)

\[ \frac{\partial \bar{E}_b}{\partial t} + \bar{U} \frac{\partial \bar{E}_b}{\partial \bar{r}} + (1 - \Gamma) \frac{\bar{U}}{\bar{r}} \bar{E}_b = - \frac{D \bar{E}_b^2}{\bar{\lambda}} + \frac{r_0}{u_0 \bar{E}_b} S \]

\[ \frac{\partial \bar{\lambda}}{\partial t} + \bar{U} \frac{\partial \bar{\lambda}}{\partial \bar{r}} + \Gamma \frac{\bar{U}}{\bar{r}} \bar{\lambda} = \frac{D \bar{E}_b^2}{2} - \frac{1}{2} \frac{I}{\bar{E}_b \bar{E}_b} \frac{r_0}{u_0} S \]

Where \( D = \frac{r_0 \sqrt{E_{b0}}}{l_0 u_0} \) is a constant \( (r_0 = 1AU, l_0 = 0.01AU, u_0 = 350km/s, E_{b0} = 540(km/s)^2) \). I have put \( D = 3.9, \bar{U} = 1 \) in the numerical solution.
Background
-The spectrum of magnetic/velocity fluctuation is kolmogorov.
Undriven Models

- No source to drive the turbulence

\[
\frac{\partial E_b}{\partial t} + \bar{U} \frac{\partial E_b}{\partial \bar{r}} + \left[ 1 - \Gamma \right] \frac{\bar{U}}{\bar{r}} E_b = -\frac{D E_b^3}{\lambda} \\
\frac{\partial \bar{\lambda}}{\partial t} + \bar{U} \frac{\partial \bar{\lambda}}{\partial \bar{r}} + \left[ \Gamma \frac{\bar{U}}{\bar{r}} \bar{\lambda} - \frac{D E_b^4}{2} \right]
\]

- The discretization of $E_b$ and $\bar{\lambda}$ based on explicit finite difference method,

\[
E_{b,n+1}^{(\bar{r})} - E_{b,n}^{(\bar{r})} - \nu \left[ E_{b,n}^{(\bar{r})} - E_{b,n-1}^{(\bar{r})} \right] + \left[ -D E_{b,n}^{\frac{3}{2}} \frac{\bar{\lambda}_n}{\bar{r}_n^2} - (1 - \Gamma) \bar{U} \frac{E_{b,n}}{\bar{r}_n^2} \right] \Delta \bar{t}
\]

\[
\bar{\lambda}_{n+1}^{(\bar{r})} - \bar{\lambda}_n^{(\bar{r})} - \nu \left[ \bar{\lambda}_n^{(\bar{r})} - \bar{\lambda}_{n-1}^{(\bar{r})} \right] + \left[ D \frac{E_{b,n}^{\frac{3}{2}}}{2} - \Gamma \frac{\bar{U}}{\bar{r}_n^2} \bar{\lambda}_n^{(\bar{r})} \right] \Delta \bar{t}
\]

Where $\Delta \bar{r}(= 0.045)$ is the step size, $\Delta \bar{t}(= 0.036)$ is the time step and $\nu = U \frac{\Delta \bar{r}}{\Delta \bar{r}} > 0$

- Steady state solution

\[
E_b = \frac{\left( \frac{1}{\bar{r}} \right)^{1-T}}{(r_0)^{2T} l_0^2 + \frac{2D}{(1+3T)} \left[ r^{1+2\bar{\lambda}} - r_0^{1+2\bar{\lambda}} \right]}
\]

\[
\bar{\lambda} = \left( \frac{1}{\bar{r}} \right)^T \left[ (r_0)^{2T} l_0^2 + \frac{2D}{(1+3T)} \left\{ r^{1+2\bar{\lambda}} - r_0^{1+2\bar{\lambda}} \right\} \right]^{\frac{1}{2}}
\]

I have assumed $r_0 - 1 = \frac{r_0}{r}$ and $l_0 - 1 = \frac{l_0}{l}$ in the numerical solution.

- Initial and boundary condition $E_b = \left( \frac{1}{\bar{r}} \right)^{\left( \frac{3}{2} \right)}$, $\bar{\lambda} = (\bar{r})^{\left( \frac{3}{2} \right)}$ for $\Gamma = 0$ and $E_b = \left( \frac{1}{\bar{r}} \right)^2$, $l = 1$ for $\Gamma = 1$. And B.C $E_b = 1$, $\bar{\lambda} = 1$
Stream Interaction Driven Models

- There is a source to drive turbulence
  
  \[ S - c_{sh} \frac{U}{r} E_b \] where \( c_{sh} \) represents the strength of the stream shear interaction

  \[
  \frac{\partial \bar{E}_b}{\partial t} + U \frac{\partial \bar{E}_b}{\partial r} + (1 - \Gamma) \frac{\bar{U}}{r} \bar{E}_b = \frac{D \bar{E}_b^2}{\bar{\lambda}} + C_{sh} \frac{\bar{U}}{r} \bar{E}_b
  \]

  \[
  \frac{\partial \bar{\lambda}}{\partial t} + U \frac{\partial \bar{\lambda}}{\partial r} + \Gamma \frac{\bar{U}}{r} \bar{\lambda} = \frac{D \bar{E}_b^2}{2} - C_{sh} \frac{\bar{U}}{2r} \bar{\lambda}
  \]

  \( C_{sh} \sim 10 \) for typical solar wind particle and \( D = \frac{m}{E_b} \cdot \frac{\sqrt{\bar{E}_b}}{u_0} \)

- The discretization of \( \bar{E}_b \) and \( \bar{\lambda} \) based on explicit finite difference method,

  \[
  E_b^{n+1} = E_b^n - \nu \left[ E_b^n - E_{b(t-1)}^n \right] + \left[ -D \frac{E_b^n}{\bar{\lambda}_t^n} - (1 - \Gamma) \bar{u}_r \frac{E_b^n}{r_t^n} + C_{sh} \bar{u}_r \frac{E_b^n}{r_t^n} \right] \Delta t
  \]

  \[
  \bar{\lambda}_t^{n+1} = \bar{\lambda}_t^n - \nu \left[ \bar{\lambda}_t^n - \bar{\lambda}_{t(t-1)}^n \right] + \left[ D \frac{\bar{E}_b^n}{r_t^n} - \Gamma \frac{\bar{u}_r}{r_t^n} \bar{\lambda}_t^n - C_{sh} \bar{u}_r \frac{\bar{\lambda}_t^n}{r_t^n} \right] \Delta t
  \]

- Steady state solution

  \[
  \bar{E}_b = \frac{\left( 1 - \Gamma - C_{sh} \right)}{1 + D F^{-1} (r_F - 1)}
  \]

  \[
  \bar{\lambda} = \left( \frac{1}{r} \right)^{\Gamma + \frac{2 \bar{E}_b}{F}} \left[ 1 + D F^{-1} (r_F - 1) \right]^{\frac{1}{2}}
  \]

  Where \( F = \frac{1 + 2 \bar{E}_b}{2 \bar{E}_b + 3r} \)
Pickup Ion Driven Models

- The source to drive turbulence is Pickup Ion

- Turbulent Transport of Normalized energy density magnetic fluctuation ($E_b$) and correlation length ($\lambda$)

\[
\frac{\partial E_b}{\partial \bar{t}} + \bar{U} \frac{\partial E_b}{\partial \bar{r}} + (1 - \Gamma) \frac{\bar{U}}{\bar{r}} E_b = -\frac{D E_b^2}{\lambda^2} + G \bar{U} \exp\left(-\frac{H}{\bar{r}}\right)
\]

\[
\frac{\partial \lambda}{\partial \bar{t}} + \bar{U} \frac{\partial \lambda}{\partial \bar{r}} + \Gamma \frac{\bar{U}}{\bar{r}} \lambda = \frac{D E_b^2}{2} - G \bar{U} \frac{\lambda}{2 E_b} \exp\left(-\frac{H}{\bar{r}}\right)
\]

Where $G = \frac{\rho_c \bar{E}_b}{E_{b0}} = 0.4$, $H = \frac{\lambda_i}{\tau_{\rho_b}} = 8$. Also, $C_{PLI} = \frac{V_{\text{PIE}}}{\tau_{\rho_b} n_{\text{esc}}}$, $\lambda_I = \frac{\lambda_i}{\tau_{\rho_b}}$

- the discretization of $E_b$ and $\lambda$ based on explicit finite difference method,

\[
E_b^{n+1} - E_b^n - \nu \left[ E_b^n - E_b^n_{(i-1)} \right] + \left[ -D \frac{E_b^2}{\lambda_i^2} - (1 - \Gamma) \frac{\bar{U}}{\tau_{\rho_b}} \frac{E_b^2}{\tau_{\rho_b}} + G \bar{U} \exp\left(-\frac{H}{\bar{r}}\right) \right] \Delta \bar{t}
\]

\[
\lambda^{n+1} - \lambda^n - \nu \left[ \lambda^n - \lambda^n_{(i-1)} \right] + \left[ D \frac{E_b^2}{\lambda_i^2} - \Gamma \frac{\bar{U}}{\tau_{\rho_b}} \lambda_i^n - G \bar{U} \frac{\lambda_i^n}{2 E_b^n} \exp\left(-\frac{H}{\bar{r}}\right) \right] \Delta \bar{t}
\]

I have used $\Delta \bar{t} = 0.0001$ and $\Delta \bar{r} = 0.045$
Undriven Models (No Mixing)

UNDRIVEN MODELS

[Graphs showing normalized energy density and normalized correlation length as functions of AU, with different lines representing various time steps.]
Undriven Models (Strong Mixing)
Stream Interaction Driven Models

(No Mixing)
Stream Interaction Driven Models
(Strong Mixing)
Pickup Ion Driven Models
(No Mixing)
Pickup Ion Driven Model
(Strong Mixing)
Summary

- Generally, the fluctuation of magnetic energy decreases with the increase of radial distance. On the other hand the correlation length increases which we can see in undriven models.
- There is a slight increase of magnetic energy density in driven model which is because of the source term.
- The correlation length in the case of pickup ion driven models decreases with increase of radial distance which is different than other two models.
- Most of the systems are in unsteady state first and as the time passes it is getting to steady state which we can see in the result.