MA614. Homework 4 (due M. Feb. 22 2016)

1. After $k-1$ steps of the Gaussian elimination process the coefficient matrix has been transformed to the form
$$A = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$
where B_{11} is $(k-1)$ by $(k-1)$ and upper triangular. Prove that A is singular if the first column of B_{22} is zero.

2. Given an $n \times n$ nonsingular matrix A, how do you efficiently solve the following problems by using Gaussian elimination with partial pivoting:

   (a) solve the linear system $A^k x = b$, where k is a positive integer, $b \in \mathbb{R}^n$.

   (b) compute $\alpha = c^T A^{-1} b$, where $b,\ c \in \mathbb{R}^n$.

   (c) solve the matrix equation $AX = B$, where $B \in \mathbb{R}^{n \times m}$.

   You should (1) describe your algorithms, (2) present them in pseudo-code (Matlab-like language. You are not required to write down the algorithm for the Gaussian elimination with partial pivoting.), and (3) give the required flops.

3. Determine whether or not each of the following matrices is symmetric positive definite without computing the eigenvalues (explain your judgement):
$$A = \begin{pmatrix} 9 & 3 & 3 \\ 3 & 10 & 5 \\ 3 & 7 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 8 \\ 4 & -4 & 1 \\ 8 & 1 & 6 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

4. Prove that if A is symmetric positive definite, then $\det(A) > 0$.

5. (a). Let A be symmetric positive definite, and suppose that $A = LDL^T$, where L is unit lower triangular and D is diagonal. Prove that the main-diagonal entries of D are all positive.

   (b). Conversely, suppose $A = LDL^T$, where L is unit lower triangular and D is diagonal. Prove that if the main-diagonal entries of D are positive, then A is positive definite.

6. If A is a nonsingular symmetric matrix, and has the factorization $A = LDM^T$, where L and M are unit lower triangular matrices and D is a diagonal matrix, show that $L = M$.