MA614 Assignment 10 (due W. April 20, 2016)

1. Let $A = \begin{bmatrix} w & x \\ x & z \end{bmatrix}$ be real and suppose that we perform the following shifted QR step:

$$\begin{aligned} A - zI &= QR \\ \bar{A} &= RQ + zI. \end{aligned}$$

Show that if $\bar{A} = \begin{bmatrix} \bar{w} & \bar{x} \\ \bar{x} & \bar{z} \end{bmatrix}$ then

$$\begin{aligned} \bar{w} &= w + x^2(w-z)/[(w-z)^2 + x^2] \\ \bar{z} &= z - x^2(w-z)/[(w-z)^2 + x^2] \\ \bar{x} &= x^3/[(w-z)^2 + x^2] \end{aligned}$$

Show the numerical result of the above shifted QR step for the following matrix:

$$A = \begin{bmatrix} 1 & 0.01 \\ 0.01 & 2 \end{bmatrix}.$$

2. (a) Let λ be a simple eigenvalue of A with right eigenvector x and left eigenvector y, normalized so that $\|x\|_2 = \|y\|_2 = 1$. Let $\lambda + \delta\lambda$ be the corresponding eigenvalue of $A + \delta A$. Show that

$$|\delta\lambda| \le s(\lambda)\|\delta A\|_2 + O(\|\delta A\|_2^2).$$

$s(\lambda) = 1/|y^* x|$ and is called the *condition number* of the eigenvalue λ.

(b) Prove that $s(\lambda) \ge 1$.

(c) Show that when A is Hermitian, $s(\lambda) = 1$.

(d) If $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$, show that the condition number of the eigenvalue a of A is

$$s(a) = \left(1 + \left(\frac{c}{a-b}\right)^2\right)^{1/2}.$$

Therefore, we conclude that if c is not small, a is close to b, then $s(a)$ is large, i.e., a is an *ill-conditioned* eigenvalue.