SECTION I: CLASSICAL AND STATISTICAL MECHANICS

I.1: Classical Mechanics: Answer 2 questions

I.2: Statistical Mechanics: Answer 1 Question

Comprehensive Examination 2011: Classical [redacted] mechanics

Section 1: Answer two questions from this section.

(1) A flexible cord of uniform density ρ and fixed length l_0 is suspended from two points of equal height, located at $(x, z) = (-a, 0)$ and $(x, z) = (a, 0)$. The acceleration due to gravity is taken to be the constant g in the negative z direction.

(a) Write the expressions for the potential energy U and the length l for a given curve $z = z(x)$ that *may* represent the shape of the hanging cord.

(b) The actual shape is defined by minimizing U for a fixed $l = l_0$. The Lagrangian for finding this minimal shape is $L = U - \lambda(l - l_0)$, where λ is arbitrary. If L is expressed as the integral of a Lagrangian density $\mathcal{L}$, *i.e.* $L = \int_{-a}^{a} \mathcal{L} dx$, show that $\mathcal{L} = \rho g\sqrt{1 + z'^2}(z + \lambda)$, and write explicitly the terms of the Euler-Lagrange equation of motion

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial z'} = \frac{\partial \mathcal{L}}{\partial z}$$

(c) Which coordinate does $\mathcal{L}$ *not* depend on? Explain why, as a consequence, the quantity $h = z'\partial\mathcal{L}/\partial z' - \mathcal{L}$ is a constant of the motion. Write this constant in terms of z, z', and λ, and show that the ensuing equation is satisfied by $z = \cosh(x/A) - \lambda$ for some constant A.

(d) The shape of the rope is completely specified once A and λ are known. Show that A is fixed by the condition $l = l_0$, resulting in the equation $2A\sinh(a/A) = l_0$. Describe qualitatively how one may determine λ.

(2) The dynamic variables of a one dimensional harmonic oscillator are q and p.

(a) Show that the integral $I = \oint pdq$ over one cycle of oscillation with total energy E is given by $I = E/\omega$.

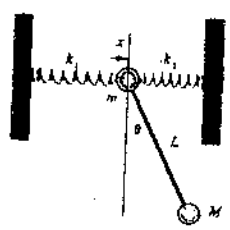
(b) A mass m slides on a frictionless horizonal track. It is connected to the wall via a spring that gradually loses its elasticity with time. Initially, the amplitude of the oscillation is A_1 and the spring constant is K_1. Assuming the integral I in (a) is constant after many cycles of oscillation, show that by the time the spring constant became K_2 the amplitude would have evolved to the value A_2, where $A_2 = A_1(K_1/K_2)^{1/4}$.

(3) One end of each of two springs with spring constants k_1 and k_2 is attached to a separate wall. A ball of mass m connects the other ends. The ball can only displace horizontally, with its equilibrium position at $x = 0$. A massless rigid rod of length l is now attached to the ball, and is free to rotate by the angle θ about a horizontal axis passing through the ball, with $\theta = 0$ occurring when the rod is vertical. Another ball of mass M is attached to the lower end of the rod.

(a) Neglecting the vertical displacement of mass m, find the Lagrangian and establish the Euler-Lagrange equations of motion for x and θ.

(b) When $m \ll M$ and the amplitude of the oscillation is small, show that $\theta \approx (k_1 + k_2)x/(Mg)$.

(c) With the help of (a) and (b), or otherwise, find the frequency of small oscillations.



Statistical Mechanics and Kinetic Theory Questions

(Choose one to answer)

1. Let N free electrons be confined to quantum states on a conducting sheet. In the corresponding two-dimensional k-space, an electron quantum state occupies an area equal to $(2\pi/L)^2$, where L is the length of the side of the sheet.

(a) Let m be the electron mass, and show that the Fermi energy, ε_F, for N electrons confined in two dimensions is given by

$$N = \frac{4\pi m}{\mathrm{h}^2}\left(\frac{L}{2\pi}\right)^2 \varepsilon_F \tag{1.1}$$

Hint: Calculate the number of electrons that can be accommodated within a circle of radius, k.

(b) Show that the density of states is given by

$$D(\varepsilon) = \frac{4\pi m}{\mathrm{h}^2}\left(\frac{L}{2\pi}\right)^2 \tag{1.2}$$

(c) Let $n = N/L^2$ be the number of electrons per unit area and use the Fermi-Dirac energy distribution

$$f(\varepsilon) = \frac{1}{\exp[\beta(\varepsilon - \mu)] + 1}$$

to show that n and the chemical potential, μ, are related by the following formula:

$$n = \frac{m}{\pi \mathrm{h}^2 \beta} \ln\left(1 + e^{\beta\mu}\right) \tag{1.3}$$

Hint: Sommerfeld's theorem does not work in this case. You should use instead the integral,

$$\int \frac{dx}{1+e^x} = -\ln\left(1+e^{-x}\right) \tag{1.4}$$

2. The classical energy of a harmonically bound particle of mass, m, is given by

$$\varepsilon = \frac{p^2}{2m} + \frac{kq^2}{2}$$

where p is the linear momentum, q is the displacement, and k is the spring constant.

(a) Let p and q have continuous values, and use the semiclassical density of states $dpdq / h$, to show that the canonical partition function is given by

$$Q = \sqrt{\frac{2m\pi k_B T}{h^2}} \sqrt{\frac{2\pi k_B T}{k}}$$

The integral

$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

will prove to be helpful.

(b) Show that the Helmholtz free energy has the form

$$A = -rT \ln(sT^v)$$

and evaluate the constants, r, s, and v.

SECTION II: ELECTROMAGNETISM AND SPECIAL RELATIVITY

II.1: Electromagnetism: Answer 2 questions

II.2: Special Relativity: Answer 1 Question

Electromagnetism

1. A large parallel plate capacitor is oriented horizontally and is filled with a linear dielectric material with permittivity $\epsilon = a + bz$, where z is the vertical distance measured from the bottom plate. The potential difference between the plates is Φ_0 (top plate is at a higher potential), the distance between the plates is d. Calculate the density of polarization charge inside the dielectric $\rho_p(z)$ and the surface density of polarization charge at the top of the dielectric σ_p. Ignore fringe effects.

2. A solid metal sphere of radius a, conductivity σ and magnetic permeability μ is placed in a uniform magnetic field slowly varying with time as

$$\mathbf{B} = B_0 \cos(\omega t)\hat{\mathbf{z}}.$$

Find (a) the magnetic field inside the sphere in the static approximation ($\omega = 0$), and (b) the eddy current density $\mathbf{J} = \sigma\mathbf{E}$ flowing in the sphere in the next approximation, by assuming that magnetic field inside the sphere found in (a) has the same dependence on time as the external field. Hint: the magnetic field inside a sphere is uniform and parallel to the external field.

3. A plane electromagnetic wave is normally incident from vacuum on a slab of dielectric with permittivity ϵ and permeability μ. The thickness of the slab is d. Find the amplitude of the reflected wave expressing your answer through the index of refraction n in the dielectric. Hint: Place the leading edge of the slab at $x = 0$ and obtain four matching conditions at the edges.

4. The magnetic energy of a localized current $\mathbf{J}$ in an external field $\mathbf{B}$ (neglecting the field produced by the current itself) is

$$U = \int (\mathbf{J} \cdot \mathbf{A}) d^3x,$$

where $\mathbf{A}$ is the vector potential of the field $\mathbf{B}$. Starting from the above equation, show that for $\mathbf{B} =$ const (uniform external field) the magnetic energy can be expressed as

$$U = -\mathbf{m} \cdot \mathbf{B},$$

where $\mathbf{m}$ is the magnetic moment associated with the current $\mathbf{J}$. Hint: align $\mathbf{B}$ with the z axis and express $\mathbf{A}$ through $\mathbf{B}$ in cylindrical coordinates.

Special Relativity Section – choose 1

1) Hyperspatial Yacht Regatta

a) In a boat race between two very well matched yachts, the faster ship gets a handicap, which means that it crosses the starting line after the first ship by a delay time, call it T, in the rest frame K. Assuming both cross the starting line at different points, separated by a distance d, for what range of T is there a frame K' where the handicap disappears? Also, for what range of T is there a 'true' handicap?

b) Determine explicitly the Lorentz transformations between K' and K for each of the cases above.

2) Angular Momentum

Given a space-time point $\mathbf{A}$, the four-vector angular momentum of a particle with momentum $\mathbf{p}$ at point $\mathbf{B}$ about $\mathbf{A}$ is given by $\mathbf{J} = \mathbf{\Delta x} \otimes \mathbf{p} - \mathbf{p} \otimes \mathbf{\Delta x}$, where $\mathbf{\Delta x}$ is the space-time separation between $\mathbf{A}$ and $\mathbf{B}$ and $\otimes$ is the tensor (or outer) product: $\mathbf{A} \otimes \mathbf{B}$ has components $A^\mu B^\nu$.

Show that for a freely moving (unaccelerated) particle, $\mathbf{J}$ is conserved; that is: $d\mathbf{J}/d\tau = 0$.

SECTION III: QUANTUM MECHANICS

Answer 3 questions

QUANTUM MECHANICS

Do any 3 of the 4 problems. Circle and clearly indicate final answers.

1. **Hamiltonians.** The quantum mechanical system "compium" is known to have exactly two stationary states, which we can represent by the orthonormal ket vectors $|1\rangle$ and $|2\rangle$. Suppose that the Hamiltonian for compium is

$$H^0 = a|1\rangle\langle 1| + b|2\rangle\langle 2| \quad .$$

(a) What kind of numbers are a and b? Explain.
(b) What are the energy eigenvalues associated with the states $|1\rangle$ and $|2\rangle$?
(c) Suppose that the Hamiltonian for compium becomes (after a suitable change to the physical system) $H = H^0 + H'$, where

$$H' = c|1\rangle\langle 2| + d|2\rangle\langle 1| \quad .$$

What is the relation between the numbers c and d?
(d) Find the eigenkets and eigenenergies for the new Hamiltonian H. (Hint: What is the matrix representation of H in the $|1\rangle$,$|2\rangle$ basis?)

2. **Spin and Spatial States.** Consider two identical spin-1/2 particles in a 1-D infinite square well extending from $x = 0$ to $x = a$. Let $E_1 = \pi^2\hbar^2/2ma^2$.
(a) Construct the total (properly symmetrized and normalized) wavefunction corresponding to a total energy $E = 2E_1$. (Neglect any interaction between the particles.)
(b) Same as part (a), but for $E = 5E_1$.

3. **Hydrogen Atom.** At time $t = 0$ a hydrogen atom is in the (spatial) state

$$\psi = \frac{1}{\sqrt{2}}\psi_{100} - \frac{i}{3\sqrt{2}}\psi_{211} + \frac{1}{3\sqrt{2}}\psi_{21-1} + \frac{\sqrt{7}}{3\sqrt{2}}\psi_{210} \quad ,$$

where $\psi_{n\ell m}$ is a usual stationary state.

(a) What values of L^2 will be found upon measurement, and with what probabilities?

(b) What is $\langle L^2 \rangle$?

(c) If no measurements are made, what is the state of the system at a later time t?

(d) If a measurement of L_z at $t = 0$ yields $\hbar$, what is the subsequent time evolution of the system?

(e) If a weak electric field is applied at $t = 0$, will there be a nonzero first-order energy perturbation?

4. **Perturbation Theory.** Consider a particle of mass m and charge q confined inside an infinite spherical well of radius R. (That is, the potential is zero for $r \leq R$ and infinity for $r > R$.)

(a) Find the ground state energy and the corresponding properly normalized wavefunction.

(b) Suppose a weak electric field $E\hat{z}$ is now applied. Find the first order shift in the ground state energy.