SECTION I: CLASSICAL AND STATISTICAL MECHANICS

I.1: Classical Mechanics: Answer 2 questions

I.2: Statistical Mechanics: Answer 1 Question
Comprehensive Examination 2011: Classical mechanics

Section 1: Answer two questions from this section.

(1) A flexible cord of uniform density \( \rho \) and fixed length \( l_0 \) is suspended from two points of equal height, located at \((x, z) = (-a, 0)\) and \((x, z) = (a, 0)\). The acceleration due to gravity is taken to be the constant \( g \) in the negative \( z \) direction.

(a) Write the expressions for the potential energy \( U \) and the length \( l \) for a given curve \( z = z(x) \) that may represent the shape of the hanging cord.

(b) The actual shape is defined by minimizing \( U \) for a fixed \( l = l_0 \). The Lagrangian for finding this minimal shape is \( L = U - \lambda (l - l_0) \), where \( \lambda \) is arbitrary. If \( L \) is expressed as the integral of a Lagrangian density \( \mathcal{L} \), i.e. \( L = \int_0^a \mathcal{L} \, dx \), show that \( \mathcal{L} = \rho g \sqrt{1 + z'^2} (z + \lambda) \), and write explicitly the terms of the Euler-Lagrange equation of motion

\[
\frac{d}{dx} \frac{\partial \mathcal{L}}{\partial z'} = \frac{\partial \mathcal{L}}{\partial z}
\]

(c) Which coordinate does \( \mathcal{L} \) not depend on? Explain why, as a consequence, the quantity \( \lambda = z' \frac{\partial \mathcal{L}}{\partial z'} - \mathcal{L} \) is a constant of the motion. Write this constant in terms of \( z, z' \), and \( \lambda \), and show that the ensuing equation is satisfied by \( z = \cosh(x/A) - \lambda \) for some constant \( A \).

(d) The shape of the rope is completely specified once \( A \) and \( \lambda \) are known. Show that \( A \) is fixed by the condition \( l = l_0 \), resulting in the equation \( 2A \sinh (a/A) = l_0 \). Describe qualitatively how one may determine \( \lambda \).

(2) The dynamic variables of a one dimensional harmonic oscillator are \( q \) and \( p \).

(a) Show that the integral \( I = \int pdq \) over one cycle of oscillation with total energy \( E \) is given by \( I = E/\omega \).

(b) A mass \( m \) slides on a frictionless horizontal track. It is connected to the wall via a spring that gradually loses its elasticity with time. Initially, the amplitude of the oscillation is \( A_1 \) and the spring constant is \( K_1 \). Assuming the integral \( I \) in (a) is constant after many cycles of oscillation, show that by the time the spring constant became \( K_2 \) the amplitude would have evolved to the value \( A_2 \), where \( A_2 = A_1 (K_1/K_2)^{1/4} \).
(3) One end of each of two springs with spring constants $k_1$ and $k_2$ is attached to a separate wall. A ball of mass $m$ connects the other ends. The ball can only displace horizontally, with its equilibrium position at $x = 0$. A massless rigid rod of length $l$ is now attached to the ball, and is free to rotate by the angle $\theta$ about a horizontal axis passing through the ball, with $\theta = 0$ occurring when the rod is vertical. Another ball of mass $M$ is attached to the lower end of the rod.

(a) Neglecting the vertical displacement of mass $m$, find the Lagrangian and establish the Euler-Lagrange equations of motion for $x$ and $\theta$.

(b) When $m \ll M$ and the amplitude of the oscillation is small, show that $\theta \approx (k_1 + k_2)x/(Mg)$.

(c) With the help of (a) and (b), or otherwise, find the frequency of small oscillations.
Statistical Mechanics and Kinetic Theory Questions

(Choose one to answer)

1. Let $N$ free electrons be confined to quantum states on a conducting sheet. In the corresponding two-dimensional $k$-space, an electron quantum state occupies an area equal to $(2\pi/L)^2$, where $L$ is the length of the side of the sheet.

(a) Let $m$ be the electron mass, and show that the Fermi energy, $\varepsilon_F$, for $N$ electrons confined in two dimensions is given by

$$N = \frac{4\pi m}{\hbar^2} \left( \frac{L}{2\pi} \right)^2 \varepsilon_F$$  \hspace{1cm} (1.1)

Hint: Calculate the number of electrons that can be accommodated within a circle of radius, $k$.

(b) Show that the density of states is given by

$$D(\varepsilon) = \frac{4\pi m}{\hbar^2} \left( \frac{L}{2\pi} \right)^2$$  \hspace{1cm} (1.2)

(c) Let $n = N/L^2$ be the number of electrons per unit area and use the Fermi-Dirac energy distribution

$$f(\varepsilon) = \frac{1}{\exp[\beta(\varepsilon - \mu)] + 1}$$

to show that $n$ and the chemical potential, $\mu$, are related by the following formula:

$$n = \frac{m}{\pi \hbar^2 \beta} \ln \left(1 + e^{\beta \mu} \right)$$  \hspace{1cm} (1.3)

Hint: Sommerfeld's theorem does not work in this case. You should use instead the integral,

$$\int \frac{dx}{1 + e^x} = - \ln \left(1 + e^{-x} \right)$$  \hspace{1cm} (1.4)
2. The classical energy of a harmonically bound particle of mass, \( m \), is given by

\[
\varepsilon = \frac{p^2}{2m} + \frac{kq^2}{2}
\]

where \( p \) is the linear momentum, \( q \) is the displacement, and \( k \) is the spring constant.

(a) Let \( p \) and \( q \) have continuous values, and use the semiclassical density of states \( dpdq/\hbar \), to show that the canonical partition function is given by

\[
Q = \sqrt{\frac{2m\pi k_b T}{\hbar^2}} \sqrt{\frac{2\pi k_b T}{k}}
\]

The integral

\[
\int_{-\infty}^{\infty} e^{-\omega t} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}
\]

will prove to be helpful.

(b) Show that the Helmholtz free energy has the form

\[
A = -RT \ln(sT^n)
\]

and evaluate the constants, \( r, s, \) and \( v \).
SECTION II: ELECTROMAGNETISM AND SPECIAL RELATIVITY

II.1: Electromagnetism: Answer 2 questions

II.2: Special Relativity: Answer 1 Question
Electromagnetism

1. A large parallel plate capacitor is oriented horizontally and is filled with a linear dielectric material with permittivity \( \epsilon = a + bz \), where \( z \) is the vertical distance measured from the bottom plate. The potential difference between the plates is \( \Phi_0 \) (top plate is at a higher potential), the distance between the plates is \( d \). Calculate the density of polarization charge inside the dielectric \( \rho_p(z) \) and the surface density of polarization charge at the top of the dielectric \( \sigma_p \). Ignore fringe effects.

2. A solid metal sphere of radius \( a \), conductivity \( \sigma \) and magnetic permeability \( \mu \) is placed in a uniform magnetic field slowly varying with time as

\[
B = B_0 \cos(\omega t) \hat{z}.
\]

Find (a) the magnetic field inside the sphere in the static approximation \( (\omega = 0) \), and (b) the eddy current density \( J = \sigma E \) flowing in the sphere in the next approximation, by assuming that magnetic field inside the sphere found in (a) has the same dependence on time as the external field. Hint: the magnetic field inside a sphere is uniform and parallel to the external field.

3. A plane electromagnetic wave is normally incident from vacuum on a slab of dielectric with permittivity \( \epsilon \) and permeability \( \mu \). The thickness of the slab is \( d \). Find the amplitude of the reflected wave expressing your answer through the index of refraction \( n \) in the dielectric. Hint: Place the leading edge of the slab at \( x = 0 \) and obtain four matching conditions at the edges.

4. The magnetic energy of a localized current \( J \) in an external field \( B \) (neglecting the field produced by the current itself) is

\[
U = \int (J \cdot A) d^3x,
\]

where \( A \) is the vector potential of the field \( B \). Starting from the above equation, show that for \( B = \text{const} \) (uniform external field) the magnetic energy can be expressed as

\[
U = -m \cdot B,
\]

where \( m \) is the magnetic moment associated with the current \( J \). Hint: align \( B \) with the \( z \) axis and express \( A \) through \( B \) in cylindrical coordinates.
Special Relativity Section – choose 1

1) Hyperspatial Yacht Regatta

a) In a boat race between two very well matched yachts, the faster ship gets a handicap, which means that it crosses the starting line after the first ship by a delay time, call it \( T \), in the rest frame \( K \). Assuming both cross the starting line at different points, separated by a distance \( d \), for what range of \( T \) is there a frame \( K' \) where the handicap disappears? Also, for what range of \( T \) is there a 'true' handicap?

b) Determine explicitly the Lorentz transformations between \( K' \) and \( K \) for each of the cases above.

2) Angular Momentum

Given a space-time point \( A \), the four-vector angular momentum of a particle with momentum \( p \) at point \( B \) about \( A \) is given by \( J = \Delta x \otimes p - p \otimes \Delta x \), where \( \Delta x \) is the space-time separation between \( A \) and \( B \) and \( \otimes \) is the tensor (or outer) product: \( A \otimes B \) has components \( A^\alpha B^\beta \).

Show that for a freely moving (unaccelerated) particle, \( J \) is conserved; that is: \( dJ/d\tau = 0 \).
SECTION III: QUANTUM MECHANICS

Answer 3 questions
QUANTUM MECHANICS

Do any 3 of the 4 problems. Circle and clearly indicate final answers.

1. Hamiltonians. The quantum mechanical system “compium” is known to have exactly two stationary states, which we can represent by the orthonormal ket vectors $|1\rangle$ and $|2\rangle$. Suppose that the Hamiltonian for compium is

$$H^0 = a|1\rangle\langle 1| + b|2\rangle\langle 2|$$

(a) What kind of numbers are $a$ and $b$? Explain.
(b) What are the energy eigenvalues associated with the states $|1\rangle$ and $|2\rangle$?
(c) Suppose that the Hamiltonian for compium becomes (after a suitable change to the physical system) $H = H^0 + H'$, where

$$H' = c|1\rangle\langle 2| + d|2\rangle\langle 1|$$

What is the relation between the numbers $c$ and $d$?
(d) Find the eigenkets and eigenenergies for the new Hamiltonian $H$. (Hint: What is the matrix representation of $H$ in the $|1\rangle,|2\rangle$ basis?)

2. Spin and Spatial States. Consider two identical spin-1/2 particles in a 1-D infinite square well extending from $x = 0$ to $x = a$. Let $E_1 = \pi^2\hbar^2/2ma^2$.

(a) Construct the total (properly symmetrized and normalized) wavefunction corresponding to a total energy $E = 2E_1$. (Neglect any interaction between the particles.)
(b) Same as part (a), but for $E = 5E_1$. 
3. **Hydrogen Atom.** At time \( t = 0 \) a hydrogen atom is in the (spatial) state

\[
\psi = \frac{1}{\sqrt{2}} \psi_{100} - \frac{i}{3\sqrt{2}} \psi_{211} + \frac{1}{3\sqrt{2}} \psi_{21-1} + \frac{\sqrt{7}}{3\sqrt{2}} \psi_{210}
\]

where \( \psi_{n\ell m} \) is a usual stationary state.

(a) What values of \( L^2 \) will be found upon measurement, and with what probabilities?

(b) What is \( \langle L^2 \rangle \)?

(c) If no measurements are made, what is the state of the system at a later time \( t \)?

(d) If a measurement of \( L_z \) at \( t = 0 \) yields \( \hbar \), what is the subsequent time evolution of the system?

(e) If a weak electric field is applied at \( t = 0 \), will there be a nonzero first-order energy perturbation?

4. **Perturbation Theory.** Consider a particle of mass \( m \) and charge \( q \) confined inside an infinite spherical well of radius \( R \). (That is, the potential is zero for \( r \leq R \) and infinity for \( r > R \).)

(a) Find the ground state energy and the corresponding properly normalized wavefunction.

(b) Suppose a weak electric field \( E \hat{z} \) is now applied. Find the first order shift in the ground state energy.