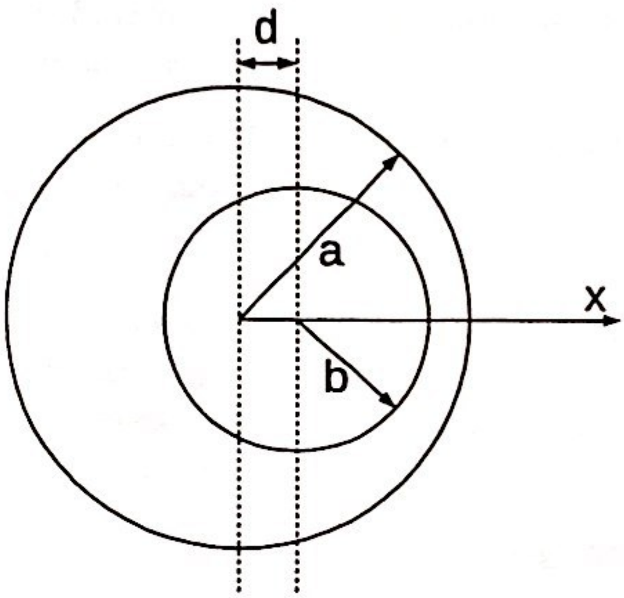
Section III:
Electromagnetism and Special Relativity

Electricity and Magnetism - problems
(Answer 2)

1. A uniform electric current with density J flows between two non-coaxial infinite cylindrical surfaces of radii a and b (see figure). The axes of the cylinders are parallel and separated by a distance $d < a - b$. Calculate the magnetic field inside the smaller cylinder. Hint: use the principle of superposition.



2. The plane $z = 0$ is kept at zero potential with the exception of the interior of a circle of radius a which is maintained at a uniform potential Φ_0. Using an expansion in Legendre polynomials, find the electric potential inside the hemisphere $r < a$, $z > 0$. Hint: seek separate solutions inside and outside the hemisphere, joining them together using boundary conditions for the potential. Remember that odd Legendre polynomials are orthogonal on the interval $[0, 1]$. You will need the following integral:

$$\int_0^1 P_{2k+1}(x)dx = \frac{(-1)^k(2k)!}{2^{2k+1}k!(k+1)!}.$$

3. Consider a cylindrical electromagnetic wave traveling in vacuum in the radial direction ($\hat{\mathbf{r}}$). Assume that the fields are independent of z and ϕ and

that both depend on time as $e^{-i\omega t}$. Using cylindrical coordinates (r, ϕ, z) show that $E_r = B_r = 0$. Next, assuming that $\mathbf{E} = E_z\hat{\mathbf{z}}$, $\mathbf{B} = B_\phi\hat{\phi}$, write down the wave equations for the fields and find their general solutions. Hint: your answer should contain Bessel functions of order 0.

Special Relativity Section – Answer 1

1) Four Velocity

a) The four-velocity is a Lorentz-invariant vector that extends the idea of regular velocity from to four dimensions. Given that the square of a four-vector is a Lorentz scalar (you should be abl to guess what it is!), determine the components of the four velocity for a general frame of referenc (one that is moving with arbitrary motion with respect to the velocity vector as measured in th observer's frame) in terms of known quantities and the three components of the 3D velocity.

b) Given that the square of the four velocity is a Lorentz scalar, determine the dot product betwee the four-velocity and four-acceleration.

c) Finally, use part b) to determine the magnitude of the instantaneous acceleration measured ir the observer's frame as an invariant.

2) Moving Mirrors

a) A mirror is moving with a velocity v perpendicular to its plane. If a photon in the lab frame is incident to the mirror with an angle θ with respect to the normal vector of the mirror, with what angle will the photon be reflected? You may assume the mirror is in the xy plane and that the photon's initial momentum in the lab frame to be:

$$p_{\text{before}} = (E, 0, E\sin\theta, E\cos\theta) \tag{1}$$

b) What is the energy of the photon in the lab frame after reflection?

c) Now suppose the mirror is moving parallel to its plane. With what angle will a similar photon as in a) be reflected? (Hint: does the azimuthal angle that the photon's path makes with respect to the mirror's velocity make any difference?)

Section II:
Quantum Mechanics

QUANTUM MECHANICS

Do any 3 of the 5 problems. Circle and clearly indicate final answers.

1. **Rotator.** Consider a rotator with moment of inertia I and dipole moment $\boldsymbol{p}$ (such as a polar molecule like H_2O).

(a) Why can't we have a quantum mechanical system where the dipole moment is constrained to rotate in a plane?

(b) Suppose the dipole moment is free to rotate in 3-D about its center of mass. What are the energy eigenvalues and eigenfunctions? Comment on degeneracy.

(c) Now suppose an electric field $\mathcal{E}\hat{z}$ is added to the system. Quantitatively describe the effect on the ground state and the first excited state.

2. **Scattering.** Consider scattering from the "hard sphere" potential $V(r) \to +\infty$ for $r \leq a$ and $V(r) = 0$ for $r > a$.

(a) Derive an expression for the partial wave phase shift δ_ℓ.

(b) Find the wavenumber k dependence of δ_ℓ in the limit of small k (low energies).

(c) Find the S-wave total cross section. Compare with the classical cross section and account for any difference.

3. **Spherical Well.** Sort of opposite to Problem 2, a particle of mass m is confined to an infinite spherical well, where $V(r) \to +\infty$ for $r \geq a$ and $V(r) = 0$ for $r < a$.

(a) Find the allowed S-state energies and properly normalized radial wavefunctions $u(r)$ (or $R(r)$ if you like).

(b) Find the pressure exerted on the well walls by the particle, if the particle is in the ground state. (Hint: First find the force. May want to think of Impulse-Momentum Theorem, or a dimensional argument.)

4. **Identical Particles.** Suppose we have a 1-D infinite square well from $x = 0$ to $x = a$.

(a) Write down the ground state and first excited state wavefunctions for two electrons in the well.

(b) If a well of width 10^{-3} m has a Fermi energy of 2 eV, how many electrons are in the well? ($\hbar c = 197 \times 10^{-6}$ eV-mm.)

(c) What is the total energy contained in these electrons?

(d) To order of magnitude, what force (in Newtons) do they exert on the well walls?

5. **Transitions.** Suppose a charge q in a 1-D simple harmonic potential $m\omega^2 x^2/2$ is initially in the state $|n\rangle$.

(a) Derive the electric dipole selection rule (or rules).

(b) Now suppose that the state decays to a state $|n'\rangle$. What is the general condition, qualitatively, that determines when spontaneous emission dominates stimulated emission in a system at thermal equilibrium with its surroundings?

Possibly Useful Information

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right)$$

Section I:
Classical Mechanics and Thermodynamics/Statistical Mechanics

Comprehensive Examination 2012: Classical and statistical mechanics

Section 1: Answer **two** questions from this section.

(1) Consider one of the rare decays of the K^+ meson of mass m_K into a π^+ of mass m_{π^+}, a π^0 of mass m_{π^0} and a photon γ of zero mass. Immediately before the decay, the K^+ is at rest. Solve the following two problems using the principle of 4-momentum conservation. You may put $c = 1$.

(a) Draw the 3-momenta of the three particles for the configuration that maximizes the energy of the π^+. What is this energy?

(b) Repeat as in (a), but for the configuration that maximizes the energy of the photon. Find the relation between the magnitudes of the 3-momenta of the π^+ and the photon.

(2) A mountain resting upon the xy plane is symmetrically peaked at the origin. In plane polar coordinates (r, θ, ϕ) where $r^2 = x^2 + y^2$, the height of the mountain is given by the function $z(r)$. A climber wishes to go from position (r_1, θ_1) to another position (r_2, θ_2) on the other side of the mountain. His/her path is specified by some function $r(\theta)$ satisfying $r(\theta_1) = r_1$ and $r(\theta_2) = r_2$.

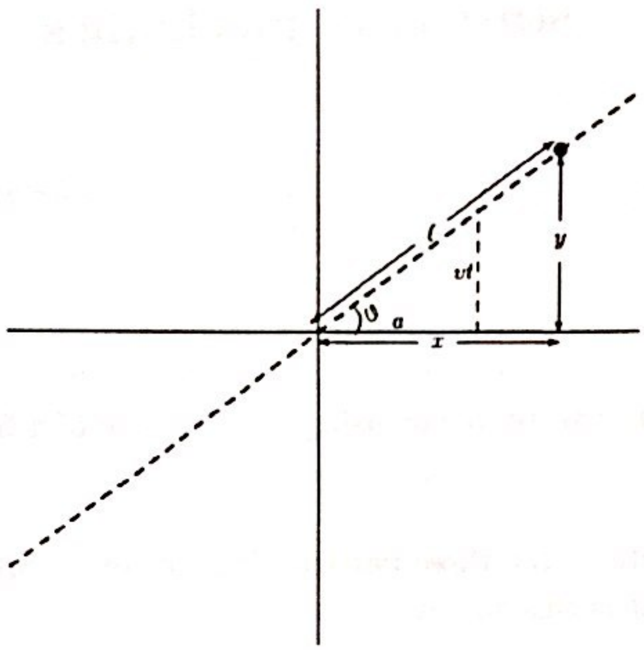
(a) Show that the length of the path may be expressed as

$$s = \int_{\theta_1}^{\theta_2} A\left(r(\theta), \dot{r}(\theta), z'(r(\theta))\right) d\theta,$$

where the dot denotes $d/d\theta$ and the prime d/dr. Write down the formula for the functional $A(r, \dot{r}, z')$.

(b) Find the equation that determines the shortest path (it is neither necessary nor useful to calculate $d/d\theta$ in this equation).

(c) Suppose the mountain is given by $z(r) = -kr$ for some constant k. Show that the path $r(\theta) = c/\cos(\alpha\theta - \varphi)$, where c and φ are arbitrary constants and $\alpha = 1/\sqrt{1+k^2}$, satisfies the equation of (b).



(3) A particle of mass m slides on a frictionless rod that rotates about the origin and on the xy plane. The angle θ of the rod, as measured from the $+x$ direction, satisfies $\tan\theta = vt/a$ for constants a and v ($a > 0$). Let ℓ be the signed distance of the mass from the origin, with ℓ defined to be positive when the x coordinate is positive. The above diagram depicts the situation for v, t, and ℓ all positive.

(a) Find the Lagrangian of the system as a function of ℓ, $\dot{\ell}$, and t.

(b) Write down the Euler-Lagrange equation of motion.

(c) Assuming (do *not* prove this) the following two expressions for ℓ:

$$\ell_1(t) = \sqrt{a^2 + v^2t^2}, \text{ and } \ell_2(t) = \sqrt{a^2 + v^2t^2}\,\arctan(vt/a)$$

are particular solutions of the equation of motion, write down the general solution. Hence or otherwise find a locus (trajectory) for which the mass is at $\ell = b$ when $t = 0$ and reaches a finite ℓ as $t \to \infty$. Find this finite value of ℓ.

Hint: for the last part of (c) you may wish to use the following series:

$$\arctan(s) = \frac{\pi}{2} - \frac{1}{s} + O(1/s^3).$$

Statistical Mechanics and Kinetic Theory Qualifier Questions

(Answer 1)

1. Consider a fictitious substance with volume independent internal energy given by $E = bT^2$, where $b = 101.3$ J/K^2, and T is the Kelvin temperature. The equation of state for this substance is $PV = aT^2$ where $a = 1$ L atm/K^2, P is the pressure, and V is the volume. Consider three states listed below:

State	P(atm)	V(L)	T(K)
1	1	10	$\sqrt{10}$
2	1	20	$\sqrt{20}$
3	0.5	20	$\sqrt{10}$

The substance is carried around the complete cycle, $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$. For each leg of the cycle, compute the change in internal energy ΔE, the change in work ΔW, and the change in heat ΔQ in Joules. Note: 1 L atm = 101.3 J. For each leg, compute the change in system entropy, ΔS, in the units of J/K. Comment: Don't worry that the system entropy does not sum to zero around the complete cycle. The sum of the entropy of the surroundings, which you are not asked to calculate, and the entropy of the system does indeed add to zero.

(a) Isobar ($1 \rightarrow 2$), compute $\Delta E, \Delta W, \Delta Q$, and ΔS.

(b) Isochor ($2 \rightarrow 3$) compute $\Delta E, \Delta W, \Delta Q$, and ΔS.

(c) Isotherm ($3 \rightarrow 1$) compute $\Delta E, \Delta W, \Delta Q$, and ΔS.

2. Let N free electrons be confined to quantum states on a conducting sheet. In the corresponding two-dimensional k-space, an electron quantum state occupies an area equal to $(2\pi / L)^2$, where L is the length of the side of the sheet.

(a) Let m be the electron mass, and show that the Fermi energy, ε_F, for N electrons confined in two dimensions is given by

$$N = \frac{4\pi m}{\mathrm{h}^2}\left(\frac{L}{2\pi}\right)^2 \varepsilon_F \tag{2.1}$$

Hint: Calculate the number of electrons that can be accommodated within a circle of radius, k.

(b) Show that the density of states is given by

$$D(\varepsilon) = \frac{4\pi m}{\mathrm{h}^2}\left(\frac{L}{2\pi}\right)^2 \tag{2.2}$$

(c) Let $n = N / L^2$ be the number of electrons per unit area and use the Fermi-Dirac energy distribution

$$f(\varepsilon) = \frac{1}{\exp[\beta(\varepsilon - \mu)] + 1}$$

to show that n and the chemical potential, μ, are related by the following formula:

$$n = \frac{m}{\pi \mathrm{h}^2 \beta} \ln\left(1 + e^{\beta\mu}\right) \tag{2.3}$$

Hint: Sommerfeld's theorem does not work in this case. You should use instead the integral,

$$\int \frac{dx}{1 + e^x} = -\ln\left(1 + e^{-x}\right) \tag{2.4}$$