

Edgeworth Expansion of the Largest Eigenvalue Distribution Function of GUE Revisited

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February 2, 2008

Abstract

We derive expansions of the resolvent $R_n(x, y; t) = (Q_n(x; t)P_n(y; t) - Q_n(y; t)P_n(x; t))/(x - y)$ of the Hermite kernel K_n at the edge of the spectrum of the finite n Gaussian Unitary Ensemble (GUE_n) and the finite n expansion of $Q_n(x; t)$ and $P_n(x; t)$. Using these large n expansions, we give another proof of the derivation of an Edgeworth type theorem for the largest eigenvalue distribution function of GUE_n . We conclude with a brief discussion on the derivation of the probability distribution function of the corresponding largest eigenvalue in the Gaussian Orthogonal Ensemble (GOE_n) and Gaussian Symplectic Ensembles (GSE_n).

1 Introduction

The author stressed in [1] the importance of having a large n -expansion of the distribution of the largest eigenvalue from classical Random Matrix Ensembles. In this paper we present another derivation of the probability distribution function of the largest eigenvalue from the GUE_n . Unlike the previous derivation which follows from the Fredholm determinant representation $\mathbb{P}(\lambda_{Max} \leq t) = \det(I - K_n)_{(t, \infty)}$, this one follows directly from the resolvent kernel representation $\mathbb{P}(\lambda_{Max} \leq t) = \exp\{-\int_t^\infty R_n(x, x; t) dx\}$. (A proof of this representation can be found in [19].) Here the Fredholm determinant expansion is replaced by the large n -expansion of P_n and Q_n . In doing this we discover new integrals relating Painlevé functions appearing in the study of the largest eigenvalue in Gaussian Ensembles. These large n -expansions can be used for the analogous problem of finding the probability distribution of the largest eigenvalue in the GOE_n and the GSE_n case.

Recall that for Gaussian Ensembles, the probability density that the eigenvalues are in infinitesimal intervals about the points x_1, \dots, x_n is given by

$$\mathbb{P}_{n\beta}(x_1, \dots, x_n) = C_{n\beta} \exp\left(-\frac{\beta}{2} \sum_1^n x_j^2\right) \prod_{j < k} |x_j - x_k|^\beta, \quad (1.1)$$

with

$$-\infty < \lambda_i < \infty, \quad \text{for } i = 1, \dots, n, \quad (1.2)$$

and $C_{n\beta}$ is the normalization constant.

Let

$$F_{n,\beta}(t) = \mathbb{P}(\lambda_{\max}^\beta \leq t) \quad (1.3)$$

be the probability distribution function of the largest eigenvalue in GOE_n for $\beta = 1$, GUE_n for $\beta = 2$, and GSE_n for $\beta = 4$ respectively.

For the Gaussian n Ensemble, the expected value of the largest eigenvalue is asymptotically $\sqrt{2n}$. Therefore as the size of the matrices grows, so does the largest eigenvalue. To have a nontrivial limit, we must center and normalize λ_{\max}^β . In doing this we keep the fine tuning constant c introduced in [1],

$$\hat{\lambda}_{\max}^\beta := \frac{\lambda_{\max}^G - (2(n+c))^{1/2}}{2^{-1/2}n^{-1/6}}. \quad (1.4)$$

To state our results we need the following definitions. Recall that if

$$\varphi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{1/2}} H_n(x) e^{-x^2/2}$$

with $H_n(x)$ the Hermite polynomials of degree n , then the Hermite kernel is

$$K_n(x, y) = \sum_{k=0}^{n-1} \varphi_k(x) \varphi_k(y).$$

The resolvent of the integral operator on $L^2(t, \infty)$ with Hermite kernel will be denoted by R_n and its kernel denoted by

$$R_n(x, y) := (I - K_n)^{-1} K_n(x, y). \quad (1.5)$$

This resolvent also has the representation (see for example [19], page 6)

$$R_n(x, y; t) = \frac{Q_n(x; t) P_n(x; t) - P_n(x; t) Q_n(y; t)}{x - y} \quad (1.6)$$

where

$$Q_{n,i}(x; t) = ((I - K_n)^{-1}, x^i \varphi_n) \quad (1.7)$$

and

$$P_{n,i}(x; t) = ((I - K_n)^{-1}, x^i \varphi_{n-1}). \quad (1.8)$$

We introduce the following quantities

$$q_{n,i}(t) = Q_{n,i}(t;t), \quad p_{n,i}(t) = P_{n,i}(t;t) \quad (1.9)$$

$$u_{n,i}(t) = (Q_{n,i}, \varphi_n), \quad v_{n,i}(t) = (P_{n,i}, \varphi_n), \quad (1.10)$$

$$\tilde{v}_{n,i}(t) = (Q_{n,i}, \varphi_{n-1}), \quad \text{and} \quad w_{n,i}(t) = (P_{n,i}, \varphi_{n-1}). \quad (1.11)$$

Here (\cdot, \cdot) denotes the inner product on $L^2(t, \infty)$. In our notation, the subscript without the n represents the scaled limit of that quantity when n goes to infinity, and we dropped the second subscript i when it is zero.

If Ai is Airy function, the kernel $K_n(x, y)$ then scales¹ to the Airy kernel

$$K_{\text{Ai}}(X, Y) = \frac{\text{Ai}(X) \text{Ai}'(Y) - \text{Ai}(Y) \text{Ai}'(X)}{X - Y}. \quad (1.12)$$

Our convention is as follow;

$$Q_i(x; s) = ((I - K_{\text{Ai}})^{-1}, x^i \text{Ai}), \quad (1.13)$$

$$P_i(x; s) = ((I - K_{\text{Ai}})^{-1}, x^i \text{Ai}'), \quad (1.14)$$

$$q_i(s) = Q_i(s; s), \quad p_i(s) = P_i(s; s), \quad (1.15)$$

$$u_i(s) = (Q_i, \text{Ai}), \quad v_i(s) = (P_i, \text{Ai}), \quad (1.16)$$

$$\tilde{v}_i(s) = (Q_i, \text{Ai}'), \quad \text{and} \quad w_i(t) = (P_i, \text{Ai}'). \quad (1.17)$$

Here (\cdot, \cdot) denotes the inner product on $L^2(s, \infty)$ and $i = 0, 1, 2$.

We use the subscript n for unscaled quantities only.

Our first result are large n -expansions of $R_n(x, y; t)$, $Q_n(x; t) = Q_{n,0}(x; t)$ and $P_n(x; t) = P_{n,0}(x; t)$.

Theorem 1.1. *For*

$$x = \sqrt{2(n+c)} + \frac{X}{2^{\frac{1}{2}}n^{\frac{1}{6}}}, \quad y = \sqrt{2(n+c)} + \frac{Y}{2^{\frac{1}{2}}n^{\frac{1}{6}}} \quad \text{and} \quad t = \sqrt{2(n+c)} + \frac{s}{2^{\frac{1}{2}}n^{\frac{1}{6}}}, \quad (1.18)$$

as $n \rightarrow \infty$ with X, Y , and s bounded,

$$\begin{aligned} R_n(x, y; t) dx = & \left[R(X, Y; s) - c Q(X; s) Q(Y; s) n^{-\frac{1}{3}} + \frac{n^{-\frac{2}{3}}}{20} \left[P_1(X; s) P(Y; s) + P(X; s) P_1(Y; s) \right. \right. \\ & - Q_2(X; s) Q(Y; s) - Q_1(X; s) Q_1(Y; s) - Q(X; s) Q_2(Y; s) + 20c^2 u_0(s) Q(X; s) Q(Y; s) \\ & \left. \left. + \frac{3 - 20c^2}{2} (P(X; s) Q(Y; s) + Q(X; s) P(Y; s)) \right] + O(n^{-1}) e_n(X, Y) \right] dX. \quad (1.19) \end{aligned}$$

The error term, $e_n(X, Y)$, is the kernel of an integral operator on $L^2(s, \infty)$ which is trace class.

¹For the precise definition of this scaling, see the next section

And

Theorem 1.2. *For*

$$x = \sqrt{2(n+c)} + \frac{X}{2^{\frac{1}{2}}n^{\frac{1}{6}}} \quad \text{and} \quad t = \sqrt{2(n+c)} + \frac{s}{2^{\frac{1}{2}}n^{\frac{1}{6}}},$$

as $n \rightarrow \infty$ with X and s bounded,

$$\begin{aligned} Q_n(x; t) &= n^{\frac{1}{6}} \left[Q(X; s) + \left[\frac{2c-1}{2} P(X; s) - cQ(X; s)u(s) \right] n^{-\frac{1}{3}} \right. \\ &+ \left[(10c^2 - 10c + \frac{3}{2})Q_1(X; s) + P_2(X; s) + (-30c^2 + 10c + \frac{3}{2})Q(X; s)v(s) \right. \\ &+ P_1(X; s)v(s) + P(X; s)v_1(s) - Q_2(X; s)u(s) - Q_1(X; s)u_1(s) - Q(X; s)u_2(s) \\ &\left. \left. + (-10c^2 + \frac{3}{2})P(X; s)u(s) + 20c^2Q(X; s)u^2(s) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1})E_q(X; s) \right], \quad (1.20) \end{aligned}$$

and

$$\begin{aligned} P_n(x; t) &= n^{\frac{1}{6}} \left[Q(X; s) + \left[\frac{2c+1}{2} P(X; s) - cQ(X; s)u(s) \right] n^{-\frac{1}{3}} \right. \\ &+ \left[(10c^2 + 10c + \frac{3}{2})Q_1(X; s) + P_2(X; s) + (-30c^2 - 10c + \frac{3}{2})Q(X; s)v(s) \right. \\ &+ P_1(X; s)v(s) + P(X; s)v_1(s) - Q_2(X; s)u(s) - Q_1(X; s)u_1(s) - Q(X; s)u_2(s) \\ &\left. \left. + (-10c^2 + \frac{3}{2})P(X; s)u(s) + 20c^2Q(X; s)u^2(s) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1})E_p(X; s) \right]. \quad (1.21) \end{aligned}$$

Theorem 1.2 together with the work of Tracy and Widom in [19], all gives another proof of the following large n -expansion of (1.3) when $\beta = 2$.

Theorem 1.3. *We set*

$$t = (2(n+c))^{\frac{1}{2}} + 2^{-\frac{1}{2}}n^{-\frac{1}{6}}s \quad \text{and} \quad (1.22)$$

$$E_{c,2}(s) = 2w_1 - 3u_2 + (-20c^2 + 3)v_0 + u_1v_0 - u_0v_1 + u_0v_0^2 - u_0^2w_0. \quad (1.23)$$

Then as $n \rightarrow \infty$

$$F_{n,2}(t) = F_2(s) \left\{ 1 + c u_0(s) n^{-\frac{1}{3}} - \frac{1}{20} E_{c,2}(s) n^{-\frac{2}{3}} \right\} + O(n^{-1}) \quad (1.24)$$

uniformly in s , and

$$F_2(s) = \lim_{n \rightarrow \infty} F_{n,2}(t) = \exp \left(- \int_s^\infty (x-s)q(x)^2 dx \right) \quad (1.25)$$

is the Tracy-Widom distribution.

In §2 we derive Theorem 1.1 and Theorem 1.2. In §3 we give another proof of (1.24) where this time we make use of the representation of the probability distribution function of the largest eigenvalue in term of the resolvent of the Hermite kernel instead of the Fredholm determinant representation using the Hermite kernel. The advantage of this derivation is that this technique also applies to the finite n GOE_n and GSE_n .

2 Large n -Expansion of $R_n(x, y; t)$, $Q_n(x; t)$ and $P_n(x; t)$

In this section we will make use of the following expansion of the Hermite kernel $K_{n,2}(x, y)$ derived in [1]. Let $\text{Ai}(x)$ be the Airy function and

$$K_{\text{Ai}}(x, y) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}(y) \text{Ai}'(x)}{x - y} = \int_0^\infty \text{Ai}(x + z) \text{Ai}(y + z) dz \quad (2.1)$$

the Airy kernel. For x and y defined by (1.18), we have as $n \rightarrow \infty$

$$K_{n,2}(x, y) dx = \left\{ K_{\text{Ai}}(X, Y) - c_G \text{Ai}(X) \text{Ai}(Y) n^{-\frac{1}{3}} + \frac{1}{20} [(X + Y) \text{Ai}'(X) \text{Ai}'(Y) - (X^2 + XY + Y^2) \text{Ai}(X) \text{Ai}(Y) + \frac{-20c_G^2 + 3}{2} (\text{Ai}'(X) \text{Ai}(Y) + \text{Ai}(X) \text{Ai}'(Y))] n^{-\frac{2}{3}} + O(n^{-1})E(X, Y) \right\} dX, \quad (2.2)$$

uniformly in s . The error term, $E(X, Y)$, is the kernel of a trace class integral operator on $L^2(s, \infty)$.

In order to simplify the notations in this paper, we treat each term appearing in (2.2) as an integral operator as well as the kernel of that operator. For example $\text{Ai}(x) \text{Ai}(y)$ will be the integral operator with this kernel.

We recall that the resolvent operator is $R_n(x, y; t) = (I - K_n)^{-1} K_n(x, y)$. We have a large n -expansion of the Hermite kernel $K_n(x, y)$, therefore to derive an expansion for the resolvent kernel we only need to derive an expansion of the kernel $(I - K_n)^{-1}(x, y)$ and multiply the two operators to have our desired result. The first part of this section will be devoted to doing that, in the second part we will use that result to derive the expansion for R_n . The third part will derive an expansion of Q_n and P_n .

2.1 Large n -Expansion of $(I - K_n)^{-1}(x, y)$

Let

$$\tau(x) = \sqrt{2(n+c)} + x 2^{-\frac{1}{2}} n^{-\frac{1}{6}} \quad (2.3)$$

be the scaling function and χ the characteristic function of the set (t, ∞) .

If L is an integral operator with kernel $L(x, y)$, we will write L_τ for the scaled integral operator with kernel

$$\frac{1}{\sqrt{2n^{\frac{1}{6}}}} L(\tau(x), \tau(y)). \quad (2.4)$$

With this convention the scaled Hermite kernel (2.2) has the following representation.

$$K_n(\tau(X), \tau(y)) d\tau(x) = \tau' K_n(\tau(X), \tau(y)) = K_{\text{Ai}}(X, Y) - c \text{Ai}(X) \text{Ai}(Y) n^{-\frac{1}{3}} +$$

$$\frac{1}{20} \left[(X + Y) \text{Ai}'(X) \text{Ai}'(Y) - (X^2 + XY + Y^2) \text{Ai}(X) \text{Ai}(Y) + \right.$$

$$\left. \frac{-20c^2 + 3}{2} (\text{Ai}'(X) \text{Ai}(Y) + \text{Ai}(X) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} + O(n^{-1})E(X, Y). \quad (2.5)$$

Following Tracy and Widom, we denote the kernel of $(I - K_{n,2})^{-1}$ by $\rho_n(x, y)$, the characteristic function of the set (t, ∞) by $\chi(x)$, and the scaled function χ_τ the characteristic function on the set (s, ∞) as $\tau(s) = t$. Note that

$$(I - K_{n,2})^{-1} = (I - \chi_\tau K_{n,2\tau} \chi_\tau)^{-1}. \quad (2.6)$$

But χ_τ scales to the characteristic function of the set (s, ∞) with $t = \tau(s)$. To simplify notations we will not mention explicitly $\chi_{(s,\infty)}$, but think of the various operators as acting on the set (s, ∞) . With this in mind, we see that the kernel of $(I - K_{n,2})^{-1}$ is

$$\rho_n(x, y) = \left(I - \frac{1}{2^{\frac{1}{2}} n^{\frac{1}{6}}} K_{n,2}(\tau(X), \tau(Y)) \right)^{-1}$$

We combine this with (2.5) to have

$$\begin{aligned} \rho_n(x, y) &= \left(I - K_{\text{Ai}}(X, Y) + c \text{Ai}(X) \text{Ai}(Y) n^{-\frac{1}{3}} - \frac{1}{20} \left[(X + Y) \text{Ai}'(X) \text{Ai}'(Y) \right. \right. \\ &\quad \left. \left. - (X^2 + XY + Y^2) \text{Ai}(X) \text{Ai}(Y) + \frac{-20c^2 + 3}{2} (\text{Ai}'(X) \text{Ai}(Y) + \text{Ai}(X) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} \right. \\ &\quad \left. + O(n^{-1})E(X, Y) \right)^{-1} \end{aligned} \quad (2.7)$$

$$\begin{aligned} &= \left((I - K_{\text{Ai}}(X, Y)) \cdot \left\{ I + (I - K_{\text{Ai}}(X, Y))^{-1} \left[c \text{Ai}(X) \text{Ai}(Y) n^{-\frac{1}{3}} - \right. \right. \right. \\ &\quad \left. \left. \frac{1}{20} \left[(X + Y) \text{Ai}'(X) \text{Ai}'(Y) - (X^2 + XY + Y^2) \text{Ai}(X) \text{Ai}(Y) + \right. \right. \right. \\ &\quad \left. \left. \left. \frac{-20c^2 + 3}{2} (\text{Ai}'(X) \text{Ai}(Y) + \text{Ai}(X) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} + O(n^{-1})E(X, Y) \right] \right\} \right)^{-1}. \end{aligned} \quad (2.8)$$

We now think of each term in the large bracket as kernel of an integral operator on (s, ∞) . We know the existence of $(I - K_{\text{Ai}})^{-1}$. If we factor out this operator in the last equation, we find that (2.8) can be represented by

$$\begin{aligned} &\left(I + (I - K_{\text{Ai}})^{-1}(X, Y) \left\{ c \text{Ai}(X) \text{Ai}(Y) n^{-\frac{1}{3}} - \frac{1}{20} \left[(X + Y) \text{Ai}'(X) \text{Ai}'(Y) \right. \right. \right. \\ &\quad \left. \left. - (X^2 + XY + Y^2) \text{Ai}(X) \text{Ai}(Y) + \frac{-20c^2 + 3}{2} (\text{Ai}'(X) \text{Ai}(Y) + \text{Ai}(X) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} \right. \right. \\ &\quad \left. \left. + O(n^{-1})E(X, Y) \right\} \right)^{-1} \cdot (I - K_{\text{Ai}})^{-1}(X, Y). \end{aligned} \quad (2.9)$$

We have the following results (see for example [22]).

If M denotes multiplication by the independent variable, then the integral operator $M^i \text{Ai} \otimes M^j \text{Ai}$ has kernel $X^i \text{Ai}(X) Y^j \text{Ai}(Y)$, the integral operator $M^i \text{Ai}' \otimes M^j \text{Ai}$ has kernel $X^i \text{Ai}'(X) Y^j \text{Ai}(Y)$, and the integral operator $M^i \text{Ai} \otimes M^j \text{Ai}'$ has kernel $X^i \text{Ai}(X) Y^j \text{Ai}'(Y)$.

If we denote by

$$\rho(X, Y) \quad \text{the kernel of the operator} \quad (I - K_{\text{Ai}})^{-1}$$

on (s, ∞) , then using representation (1.13) we find that the kernel of

$$(I - K_{\text{Ai}})^{-1} \cdot M^i \text{Ai} \otimes M^j \text{Ai} \quad (2.10)$$

(the dot here represent operator multiplication) is

$$(\rho(X, Z), Z^i \text{Ai}(Z))_{(s, \infty)} Y^j \text{Ai}(Y) = Q_i(X; s) Y^j \text{Ai}(Y), \quad (2.11)$$

the kernel of

$$(I - K_{\text{Ai}})^{-1} \cdot M^i \text{Ai}' \otimes M^j \text{Ai} \quad (2.12)$$

is

$$(\rho(X, Z), Z^i \text{Ai}'(Z))_{(s, \infty)} Y^j \text{Ai}(Y) = P_i(X; s) Y^j \text{Ai}(Y), \quad (2.13)$$

and the kernel of

$$(I - K_{\text{Ai}})^{-1} \cdot M^i \text{Ai}' \otimes M^j \text{Ai}' \quad (2.14)$$

is given by

$$(\rho(X, Z), Z^i \text{Ai}'(Z))_{(s, \infty)} Y^j \text{Ai}'(Y) = P_i(X; s) Y^j \text{Ai}'(Y). \quad (2.15)$$

If we substitute these results in (2.9), we have

$$\begin{aligned} \rho_n(x, y) = & \left(I - \left\{ -cQ(X; s) \text{Ai}(Y) n^{-\frac{1}{3}} + \frac{1}{20} \left[P_1(X; s) \text{Ai}'(Y) + P(X; s) Y \text{Ai}'(Y) \right. \right. \right. \\ & - Q_2(X; s) \text{Ai}(Y) - Q_1(X; s) Y \text{Ai}(Y) - Q(X; s) Y^2 \text{Ai}(Y) + \frac{-20c^2 + 3}{2} P(X; s) \text{Ai}(Y) \\ & \left. \left. \left. + \frac{-20c^2 + 3}{2} Q(X; s) \text{Ai}'(Y) \right] n^{-\frac{2}{3}} + O(n^{-1}) E(X, Y) \right\} \right)^{-1} \cdot (I - K_{\text{Ai}})^{-1}(X, Y). \end{aligned} \quad (2.16)$$

We keep the same notation for the error term which is still a trace class operator since the product of the bounded operator $(I - K_{\text{Ai}})^{-1}$ with the trace class operator E is trace class.

Note that the operator $n^{-\frac{1}{3}}L$ in the braces in (2.16) is a finite sum of finite rank operators, and therefore a trace class operator. We have the following representation of our scaled operator

$$\left(I - K_{n,2} \right)^{-1} = \left(I - n^{-\frac{1}{3}}L \right)^{-1} \cdot \left(I - K_{\text{Ai}} \right)^{-1}. \quad (2.17)$$

The trace class limit of the first factor on the right is the identity operator which is invertible. Then for large n we can assume that $(I - n^{-\frac{1}{3}}L)$ is also invertible. This operator therefore admits a convergent (in trace class norm) Neumann series expansion for large n of the form

$$\left(I - n^{-\frac{1}{3}}L \right)^{-1} = \sum_{k=0}^{\infty} n^{-\frac{k}{3}}L^k = I + n^{-\frac{1}{3}}L + n^{-\frac{2}{3}}L^2 + O(n^{-1})E(X, Y). \quad (2.18)$$

We need to find a large n -expansion of $n^{-\frac{2}{3}}L^2$.

$$\begin{aligned} n^{-\frac{2}{3}}L^2 = & \left\{ -cQ(X; s) \text{Ai}(Y)n^{-\frac{1}{3}} + \frac{1}{20} \left[P_1(X; s) \text{Ai}'(Y) + P(X; s)Y \text{Ai}'(Y) \right. \right. \\ & \left. \left. - Q_2(X; s) \text{Ai}(Y) - Q_1(X; s)Y \text{Ai}(Y) - Q(X; s)Y^2 \text{Ai}(Y) + \frac{-20c^2 + 3}{2}P(X; s) \text{Ai}(Y) \right. \right. \\ & \left. \left. + \frac{-20c^2 + 3}{2}Q(X; s) \text{Ai}'(Y) \right] n^{-\frac{2}{3}} + O(n^{-1})E(X, Y) \right\}^2. \end{aligned} \quad (2.19)$$

If we use the representation (1.10), we find that this square is

$$\begin{aligned} & (-cQ(X; s) \text{Ai}(Y)n^{-\frac{1}{3}}) \cdot (-cQ(X; s) \text{Ai}(Y)n^{-\frac{1}{3}}) + O(n^{-1})E_1(X, Y) \\ & = c^2Q(X; s) (\text{Ai}(Z), Q(Z, s))_{(s, \infty)} \text{Ai}(Y) n^{-\frac{2}{3}} + O(n^{-1})E_1(X, Y) \\ & = c^2Q(X; s) u(s) \text{Ai}(Y) n^{-\frac{2}{3}} + O(n^{-1})E_1(X, Y). \end{aligned} \quad (2.20)$$

If we substitute (2.20) in the series expansion (2.18), we find that (2.16) becomes

$$\begin{aligned} \tau' \rho_n(\tau(X), \tau(Y)) = & \left(I - cQ(X; s) \text{Ai}(Y)n^{-\frac{1}{3}} + \frac{1}{20} \left[P_1(X; s) \text{Ai}'(Y) + P(X; s)Y \text{Ai}'(Y) \right. \right. \\ & \left. \left. - Q_2(X; s) \text{Ai}(Y) - Q_1(X; s)Y \text{Ai}(Y) - Q(X; s)Y^2 \text{Ai}(Y) + \frac{-20c^2 + 3}{2}P(X; s) \text{Ai}(Y) \right. \right. \\ & \left. \left. + \frac{-20c^2 + 3}{2}Q(X; s) \text{Ai}'(Y) + 20c^2Q(X; s) u(s) \text{Ai}(Y) \right] n^{-\frac{2}{3}} \right) \cdot (I - K_{\text{Ai}})^{-1}(X, Y). \\ & + O(n^{-1})E(X, Y) \cdot (I - K_{\text{Ai}})^{-1}(X, Y). \end{aligned} \quad (2.21)$$

The notation used for the error term suggests that at each step of the expansion, we add to the existing error term all the $O(n^{-1})$ -terms and rename the error term by $E(X, Y)$. This is our first result which we restate as the following Lemma.

Lemma 2.1. *Let ρ_n be the kernel of the operator $(I - K_{n,2})^{-1}$ on (t, ∞) , and τ the transformation defined by (2.3). Then as $n \rightarrow \infty$ with $x = \tau(X)$ and $y = \tau(Y)$,*

$$\rho_n(x, y) = \left(I - cQ(X; s) \text{Ai}(Y)n^{-\frac{1}{3}} + \frac{1}{20} \left[P_1(X; s) \text{Ai}'(Y) + P(X; s)Y \text{Ai}'(Y) \right. \right.$$

$$\begin{aligned}
& -Q_2(X; s) \text{Ai}(Y) - Q_1(X; s) Y \text{Ai}(Y) - Q(X; s) Y^2 \text{Ai}(Y) + \frac{-20c^2 + 3}{2} P(X; s) \text{Ai}(Y) \\
& + \frac{-20c^2 + 3}{2} Q(X; s) \text{Ai}'(Y) + 20c^2 Q(X; s) u(s) \text{Ai}(Y) \Big] n^{-\frac{2}{3}} \Big) (I - K_{\text{Ai}})^{-1}(X, Y). \\
& + O(n^{-1}) E(X, Y)
\end{aligned} \tag{2.22}$$

uniformly in s .

The error term E is the kernel of a trace class operator on (s, ∞) . Here $P(X, s) = P_0(X, s)$, $Q(X, s) = Q_0(X, s)$, Q_1 , P_1 , and Q_2 are defined in (1.13) and (1.14)

2.2 Large n -expansion of $R_n(x, y)$

In this section we will combine Lemma 2.1 and (1.6) to derive an expansion of $R_n(x, y)$. (1.6) says that

$$\tau' R_n(\tau(X), \tau(Y)) = \left(\rho_n(\tau(X), \tau(Z)), \tau' K_{n,2}(\tau(Z), \tau(Y)) \right)_{(s, \infty)} \tag{2.23}$$

First the action of $(I - K_{\text{Ai}})^{-1}$ on (2.5) gives

$$\begin{aligned}
& R(X, Y) - cQ(X; s) \text{Ai}(Y) n^{-\frac{1}{3}} + \frac{1}{20} \left[P_1(X; s) \text{Ai}'(Y) + P(X; s) Y \text{Ai}'(Y) \right. \\
& \quad \left. - Q_2(X; s) \text{Ai}(Y) - Q_1(X; s) Y \text{Ai}(Y) - Q(X; s) Y^2 \text{Ai}(Y) + \right. \\
& \quad \left. \frac{-20c^2 + 3}{2} (P(X; s) \text{Ai}(Y) + Q(X; s) \text{Ai}'(Y)) \right] n^{-\frac{2}{3}} + O(n^{-1}) E(X, Y).
\end{aligned} \tag{2.24}$$

Next the action of the first factor in (2.22) can be computed as follow:

The identity will reproduce (2.24), the $n^{-\frac{1}{3}}$ term will contribute

$$-cQ(X; s) (\text{Ai}(Z), R(Z, Y)) n^{-\frac{1}{3}} + c^2 Q(X; s) (\text{Ai}(Z), Q(X; s)) \text{Ai}(Y) n^{-\frac{2}{3}}, \tag{2.25}$$

and the $n^{-\frac{2}{3}}$ term will contribute

$$\begin{aligned}
& \frac{1}{20} \left[P_1(X; s) (\text{Ai}'(Z), R(X, Y)) + P(X; s) (Z \text{Ai}'(Z), R(Z, Y)) - Q_2(X; s) (\text{Ai}(Z), R(Z, Y)) \right. \\
& \quad \left. - Q_1(X; s) (Z \text{Ai}(Z), R(Z, Y)) - Q(X; s) (Z^2 \text{Ai}(Z), R(Z, Y)) + \right. \\
& \quad \left. + \frac{-20c^2 + 3}{2} \left(P(X; s) (\text{Ai}(Z), R(Z, Y)) + Q(X; s) (\text{Ai}'(Z), R(Z, Y)) \right) \right. \\
& \quad \left. + 20c^2 Q(X; s) u(s) (\text{Ai}(Z), R(Z, Y)) \right] n^{-\frac{2}{3}}.
\end{aligned} \tag{2.26}$$

To evaluate the various inner-product appearing in (2.25) and (2.26) we will make use of the following representation $R(X, Y) = \rho(X, Y) - \delta(X - Y)$. Thus

$$\begin{aligned} (\text{Ai}(Z), R(Z, Y)) &= -\text{Ai}(Y) + Q(Y; s), \\ (\text{Ai}'(Z), R(Z, Y)) &= -\text{Ai}'(Y) + P(Y; s), \\ (Z \text{Ai}(Z), R(Z, Y)) &= -Y \text{Ai}(Y) + Q_1(Y; s), \\ (Z^2 \text{Ai}(Z), R(Z, Y)) &= -Y^2 \text{Ai}(Y) + Q_2(Y; s), \\ (Z \text{Ai}'(Z), R(Z, Y)) &= -Y \text{Ai}'(Y) + P_1(Y; s). \end{aligned}$$

We substitute these values in (2.25) and (2.26), then add all the contributions from (2.24), (2.25) and (2.26) to obtain Theorem 1.1.

2.3 Large n -Expansion of $Q_n(x; t)$ and $P_n(x; t)$

In this section we will use Lemma 2.1 together with the expansion of $\varphi_n(x)$ and $\varphi_{n-1}(x)$ derived² in [1] to give a large n -expansion of $Q_n(x; t)$ and $P_n(x; t)$ defined by (1.7) and (1.8) respectively. To obtain $\varphi_n(x)$ from Theorem 1.1 of [1], we need to make the substitution $c \rightarrow c + \frac{1}{2}$ and the factor in Theorem 1.1 is now $n^{\frac{1}{6}}$. For $\varphi_{n-1}(x)$, we need to make the substitution $c \rightarrow c + \frac{3}{2}$ and the factor is now $n^{\frac{1}{6}} [1 + 2^{-1} n^{-\frac{2}{3}} X]$. We assume without lost of generalities that $n = 2k$ is even, and for L_k^α the Laguerre polynomial of degree k and order $\alpha > -1$ we define

$$\varphi_n(x) = \left(\frac{n}{2}\right)^{\frac{1}{4}} \frac{H_n(x) e^{-x^2/2}}{(2^n n! \sqrt{\pi})^{1/2}} = \frac{k^{\frac{1}{4}} (-1)^k 2^{2k} (k!) L_k^{-\frac{1}{2}}(x^2) e^{-x^2/2}}{(2^{2k} (2k)! \sqrt{\pi})^{1/2}}$$

and

$$\varphi_{n-1}(x) = \left(\frac{n}{2}\right)^{\frac{1}{4}} \frac{H_{n-1}(x) e^{-x^2/2}}{(2^{n-1} (n-1)! \sqrt{\pi})^{1/2}} = \frac{k^{\frac{1}{4}} (-1)^{k-1} 2^{2k-1} (k-1)! x L_{k-1}^{\frac{1}{2}}(x^2) e^{-x^2/2}}{(2^{2k-2} (2k-2)! \sqrt{\pi})^{1/2}}.$$

We then have for $x = \tau(X)$,

$$\begin{aligned} \varphi_n(x) = n^{\frac{1}{6}} \left\{ \text{Ai}(X) + \frac{(2c-1)}{2} \text{Ai}'(X) n^{-\frac{1}{3}} + \left[(10c^2 - 10c + \frac{3}{2}) X \text{Ai}(X) \right. \right. \\ \left. \left. + X^2 \text{Ai}'(X) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) \text{Ai}(X) \right\} \end{aligned} \quad (2.27)$$

and

$$\varphi_{n-1}(x) = n^{\frac{1}{6}} \left\{ \text{Ai}(X) + \frac{(2c+1)}{2} \text{Ai}'(X) n^{-\frac{1}{3}} + \left[(10c^2 + 10c + \frac{3}{2}) X \text{Ai}(X) \right. \right.$$

²This is the direct consequence of Theorem 1.1 of [1]

$$+ X^2 \text{Ai}'(X) \left] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) \text{Ai}(X) \right\} \quad (2.28)$$

Next we apply the scaled operator $(I - K_n)^{-1}$ to these functions to have our stated result. First

$$(I - K_{\text{Ai}})^{-1} \varphi_n(\tau(X)) = n^{\frac{1}{6}} \left\{ Q(X; s) + \frac{(2c-1)}{2} P(X; s) n^{-\frac{1}{3}} + \right.$$

$$\left. \left[(10c^2 - 10c + \frac{3}{2}) Q_1(X; s) + P_2(X; s) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) Q(X; s) \right\},$$

and

$$(I - K_{\text{Ai}})^{-1} \varphi_{n-1}(\tau(X)) = n^{\frac{1}{6}} \left\{ Q(X; s) + \frac{(2c+1)}{2} P(X; s) n^{-\frac{1}{3}} + \right.$$

$$\left. \left[(10c^2 - 10c + \frac{3}{2}) Q_1(X; s) + P_2(X; s) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) Q(X; s) \right\}.$$

Proceeding in a similar fashion as in the derivation of R_n , we make the first factor in the right of the (2.22) acts on these last two functions and have Theorem 1.2. Note that the inner products here are of the form

$$(Z^i \text{Ai}(Z), Q(Z, s)) = u_i(s) \quad \text{and} \quad (Z^i \text{Ai}'(Z), Q(Z, s)) = v_i(s).$$

To conclude this section, we give the following consequence of Theorem 1.2. If we set $t = \tau(s)$ then as $n \rightarrow \infty$

$$\begin{aligned} q_n(\tau(s)) &= Q_n(\tau(s); \tau(s)) = n^{\frac{1}{6}} \left(q(s) + \left[\frac{2c-1}{2} p(s) - cq(s)u(s) \right] n^{\frac{1}{3}} \right. \\ &\quad + \left[(10c^2 - 10c + \frac{3}{2}) q_1(s) + p_2(s) + (-30c^2 + 10c + \frac{3}{2}) q(s)v(s) \right. \\ &\quad \left. + p_1(s)v(s) + p(s)v_1(s) - q_2(s)u(s) - q_1(s)u_1(s) - q(s)u_2(s) \right. \\ &\quad \left. + (-10c^2 + \frac{3}{2}) p(s)u(s) + 20c^2 q(s)u^2(s) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) e_q(s) \Big), \end{aligned} \quad (2.29)$$

and

$$\begin{aligned} p_n(\tau(s)) &= P_n(\tau(s); \tau(s)) = n^{\frac{1}{6}} \left(q(s) + \left[\frac{2c+1}{2} p(s) - cq(s)u(s) \right] n^{\frac{1}{3}} \right. \\ &\quad + \left[(10c^2 + 10c + \frac{3}{2}) q_1(s) + p_2(s) + (-30c^2 - 10c + \frac{3}{2}) q(s)v(s) \right. \\ &\quad \left. + p_1(s)v(s) + p(s)v_1(s) - q_2(s)u(s) - q_1(s)u_1(s) - q(s)u_2(s) \right. \\ &\quad \left. + (-10c^2 + \frac{3}{2}) p(s)u(s) + 20c^2 q(s)u^2(s) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1}) e_p(s) \Big) \end{aligned} \quad (2.30)$$

Uniformly in s . We use the notation

$$q_i(s) = Q_i(s; s), \quad p_i(s) = P_i(s; s), \quad e_q(s) = E_Q(s; s) \quad \text{and} \quad e_p(s) = E_P(s; s)$$

and the subscript n is reserved for functions depending on the size n of the matrices in consideration.

3 Large n -Expansion of $F_{n,2}(t)$

In this section we will use the following Fredholm determinant representation of the probability distribution function of the largest eigenvalue $F_{n,2}(t)$ in the GUE_n case:

$$F_{n,2}(t) = \mathbb{P}(\lambda_{Max} \leq t) = \det(I - K_{n,2}). \quad (3.1)$$

We also have the following two equations, the proof of which can be found in [19]

$$\frac{\partial}{\partial t} \log \det(I - K_{n,2}) = -R_n(t, t; t), \quad (3.2)$$

$$\frac{\partial}{\partial t} R_n(t, t; t) = -2q_n(t)p_n(t). \quad (3.3)$$

Equation (3.3) gives

$$\frac{\partial}{\partial t} \log \det(I - K_{n,2}) = -2 \int_t^\infty q_n(x)p_n(x) dx$$

where we used the boundary conditions $(q_n p_n)(\infty) = 0$.

Integration by parts and another use of the boundary conditions gives

$$\log \det(I - K_{n,2}) = -2 \int_t^\infty \left(\int_y^\infty q_n(x)p_n(x) dx \right) dy = -2 \int_t^\infty (x-t)q_n(x)p_n(x) dx; \quad (3.4)$$

and hence

Theorem 3.1.

$$F_{n,2}(t) = \det(I - K_{n,2}) = \exp \left(-2 \int_t^\infty (x-t)q_n(x)p_n(x) dx \right). \quad (3.5)$$

Observe that this is the finite n analogue of (1.25).

Now we set $t = \tau(s)$ and $x = \tau(X)$, then

$$2 \int_t^\infty (x-t)q_n(x)p_n(x) dx = \int_s^\infty (X-s) \frac{1}{n^{\frac{1}{3}}} q_n(\tau(X)) p_n(\tau(X)) dX$$

With the help of (2.29) and (2.30), the integrand in this last equation is now

$$(x-s) \frac{1}{n^{\frac{1}{3}}} q_n(\tau(x)) p_n(\tau(x)) =$$

$$\begin{aligned}
f(x; s) = (x - s) & \left\{ q^2(x) + 2c \left[p(x)q(x) - u(x)q^2(x) \right] n^{-\frac{1}{3}} + \left[(20c^2 + 3)q(x)q_1(x) + \right. \right. \\
& 2p_2(x)q(x) + (-60c^2 + 3)q^2(x)v(x) + 2p_1(x)q(x)v(x) + 2p(x)q(x)v_1(x) - 2q_2(x)q(x)u(x) \\
& - 2q_1(x)q(x)u_1(x) - 2q^2(x)u_2(x) + (-60c^2 + 3)p(x)q(x)u(x) + 60c^2q^2(x)u^2(x) \\
& \left. \left. + (20c^2 - 5)p^2(x) \right] \frac{n^{-\frac{2}{3}}}{20} + O(n^{-1})e(x) \right\}, \tag{3.6}
\end{aligned}$$

or

$$f(x; s) = (x - s) \left(q^2(x) + a(x)n^{-\frac{1}{3}} + b(x)\frac{n^{-\frac{2}{3}}}{20} + O(n^{-1})e(x) \right).$$

where

$$a(x) = 2c[p(x)q(x) - u(x)q^2(x)] \tag{3.7}$$

and

$$\begin{aligned}
b(x) = & \left[(20c^2 + 3)q(x)q_1(x) + 2p_2(x)q(x) + (-60c^2 + 3)q^2(x)v(x) + 2p_1(x)q(x)v(x) \right. \\
& + 2p(x)q(x)v_1(x) - 2q_2(x)q(x)u(x) - 2q_1(x)q(x)u_1(x) - 2q^2(x)u_2(x) + 60c^2q^2(x)u^2(x) \\
& \left. + (-60c^2 + 3)p(x)q(x)u(x) + (20c^2 - 5)p^2(x) \right]. \tag{3.8}
\end{aligned}$$

We use x instead of X here to simplify notation since X is just a variable of integration. We therefore have

$$\begin{aligned}
\det(I - K_{n,2}) &= \exp \left(- \int_s^\infty (x - s) \left(q^2(x) + a(x)n^{-\frac{1}{3}} + b(x)\frac{n^{-\frac{2}{3}}}{20} + O(n^{-1})e(x) \right) dx \right) = \\
& \exp \left(- \int_s^\infty (x - s) q^2(x) dx \right) \exp \left(- \int_s^\infty (x - s) a(x) dx n^{-\frac{1}{3}} \right) \exp \left(- \int_s^\infty (x - s) b(x) dx \frac{n^{-\frac{2}{3}}}{20} \right) \cdot \\
& \exp \left(- \int_s^\infty (x - s) e(x) dx O(n^{-1}) \right) = \\
& \left(1 - \int_s^\infty (x - s) a(x) dx n^{-\frac{1}{3}} + \frac{1}{2} \left[\int_s^\infty (x - s) a(x) dx \right]^2 n^{-\frac{2}{3}} + E_a(s) O(n^{-1}) \right) \cdot \\
& \left(1 - \int_s^\infty (x - s) b(x) dx \frac{n^{-\frac{2}{3}}}{20} + E_b(s) O(n^{-1}) \right) \cdot \\
& \left(1 - E_e(x) O(n^{-1}) \right) \exp \left(- \int_s^\infty (x - s) q^2(x) dx \right) \\
& = \left\{ 1 - \int_s^\infty (x - s) a(x) dx n^{-\frac{1}{3}} + \left(10 \left[\int_s^\infty (x - s) a(x) dx \right]^2 - \int_s^\infty (x - s) b(x) dx \right) \frac{n^{-\frac{2}{3}}}{20} \right. \\
& \left. + E_F(s) O(n^{-1}) \right\} \exp \left(- \int_s^\infty (x - s) q^2(x) dx \right),
\end{aligned}$$

where $E_a(s)$, $E_b(s)$, and $E_e(s)$ are the remainder when expanding the exponential functions and $E_F(s)$ is the collection of all the $O(n^{-1})$ terms. The second factor of this last equality is known as the Tracy-Widom distribution. The first factor will be the focus on the remainder of this paper. First we will find a simplification for the $n^{-\frac{1}{3}}$ term, then a simplification of the $n^{-\frac{2}{3}}$ term. The error term is a consequence of our asymptotic.

3.1 The $n^{-\frac{1}{3}}$ term

We will use the displayed formulas on page 6 of [22] to simplify this factor. First note that if we integrate by parts and use the boundary conditions on $a(x)$,

$$\int_s^\infty (x-s)a(x)dx = \int_s^\infty \left(\int_y^\infty a(x) dx \right) dy. \quad (3.9)$$

We have

$$\begin{aligned} - \int_s^\infty \left(\int_y^\infty a(x) dx \right) dx &= -c \int_s^\infty \left(\int_y^\infty 2q(x)(p(x) - u(x)q(x)) dx \right) dy \\ &= -c \int_s^\infty \left(\int_y^\infty 2q(x)q'(x) dx \right) dy = -c \int_s^\infty \left(\int_y^\infty (q^2(x))' dx \right) dy \\ &= -c \int_s^\infty (-q^2(y)) dy = -c \int_s^\infty u'(y) dy = cu(s). \end{aligned} \quad (3.10)$$

Note that this is the $n^{-\frac{1}{3}}$ term from [1].

3.2 The $n^{-\frac{2}{3}}$ term

We will simplify $b(x)$ in two steps. In the first part we will simplify the expression containing the constant c , and in the second step simplify the remaining expression.

The expression proportional to the constant c is

$$10 \left[\int_s^\infty (x-s)a(x)dx \right]^2 - 20c^2 \int_s^\infty (x-s)(qq_1 - 3q^2v + 3q^2u^2 - 3pqu + p^2)(x)dx \quad (3.11)$$

Equation (3.10) says that the first term is $10c^2u^2(s)$. Equation (2.12) of [19] together with our our definition of q_i give $q_1(s) = sq(s) - v(s)q(s) + u(s)p(s)$. If we substitute this expression of $q_1(s)$ in the second term of (3.11), then it becomes

$$\begin{aligned} &\int_s^\infty (x-s)(xq^2(x) - 4q^2(x)v(x) + 3q^2(x)u^2(x) - 2p(x)q(x)u(x) + p^2(x))dx \\ &= \int_s^\infty \int_y^\infty (xq^2(x) - 4q^2(x)v(x) + 3q^2(x)u^2(x) - 2p(x)q(x)u(x) + p^2(x))dx dy. \end{aligned}$$

In the following steps we integrate this last expression

$$xq^2(x) - 4q^2(x)v(x) + 3q^2(x)u^2(x) - 2p(x)q(x)u(x) + p^2(x)$$

$$\begin{aligned}
&= -2u(x)p(x)q(x) + 2u^2(x)q^2(x) + u^2(x)q^2(x) - 2v(x)q^2(x) - 2v(x)q^2(x) + xq^2(x) \\
&\quad - 2v(x)q^2(x) + xq^2(x) + u(x)p(x)q(x) - u(x)p(x)q(x) + p^2(x) \\
&= -2(p(x)-q(x)u(x))q(x)u(x) + q^2(x)(u^2(x)-2v(x)) + q(x)(xq(x)-2v(x)q(x)+p(x)u(x)) \\
&\quad + q(x)(xq(x) - 2v(x)q(x) + p(x)u(x)) + p(x)(p(x) - q(x)u(x)) \\
&\quad = -2q'(x)q(x)u(x) + q^2(x)q^2(x) + q(x)p'(x) + p(x)q'(x) \\
&= (-q^2(x))'u(x) - q^2(x)u'(x) + (p(x)q(x))' = (-q^2(x)u(x))' + (p(x)q(x))' \\
&\quad = \left(\frac{1}{2}u^2(x)\right)'' - v''(x). \tag{3.12}
\end{aligned}$$

The second integral in (3.11) is therefore,

$$20c^2v(s) - 10c^2u^2(s). \tag{3.13}$$

This last expression is due to the fact that the functions u_i , u'_i , v_i and v'_i are zero at infinity. The derivation of the various integrals used for (3.12) can be found in [22]. We showed that the term containing the constant c simplifies to

$$20c^2v(s). \tag{3.14}$$

Note that this is the same term derived in [1].

In similar way we show that

$$\begin{aligned}
&\int_s^\infty \int_y^\infty (-3q_1q + 3q^2v - 3p^2 + 3upq)(x) dx dy \\
&= \int_s^\infty \int_y^\infty (-3xq^2 + 6q^2v - 3upq + 3upq - 3p^2)(x) dx dy \\
&= \int_s^\infty \int_y^\infty (-3q[xq - 2q^2v + up] - 3p[-uq + p])(x) dx dy \\
&= -3 \int_s^\infty \int_y^\infty (qp' + pq')(x) dx dy = 3 \int_s^\infty \int_y^\infty (v(x))'' dx dy = 3v(s). \tag{3.15}
\end{aligned}$$

Suppose that L is the integral of l subject to the boundary condition $L(\infty) = 0$, then

$$-\int_s^\infty (x-s)l(x) dx = \int_s^\infty L(x) dx. \tag{3.16}$$

Using the following representation (the derivation of which can be found in [19],)

$$(-q_1q + q^2v - p^2 + upq)(x) = (u_1 - uv + w)'(x),$$

we find that equation (3.15) reduces to the following integral

$$\int_s^\infty (u_1 - uv + w)(x) dx = -v(s), \tag{3.17}$$

or that

$$u_1(s) - u(s)v(s) + w(s) = v'(s) = -p(s)q(s). \quad (3.18)$$

At this stage of the simplification the $n^{-\frac{2}{3}}$ term is

$$\begin{aligned} (20c^2 - 3)v(s) - \int_s^\infty (x-s)(6qq_1 + 2p_2q + 2p_1qv + 2pqv_1 - 2q_2qu - 2q_1qu_1 - 2q^2u_2 - 2p^2)(x) dx \\ = (20c^2 - 3)v(s) - \int_s^\infty (x-s)h(x) dx. \end{aligned}$$

If we note that

$$\begin{aligned} h(x) &= (-6u_1' - 2v_2' - 2v_1'v - 2v_1v_1' + 2u_2'u + (u_1^2)' + 2u'u_2 + 2w')(x) \\ &= (-6u_1 - 2v_2 - 2v_1v + 2u_2u + u_1^2 + 2w)'(x), \end{aligned}$$

then the $n^{-\frac{2}{3}}$ term becomes

$$(20c^2 - 3)v(s) + \int_s^\infty (-6u_1 - 2v_2 - 2v_1v + 2u_2u + u_1^2 + 2w)(x) dx. \quad (3.19)$$

This is where we stop our simplification of this term. To fully simplify this result to match equation (1.24), we need to derive³ new integral similar to equation (3.18) from the following representation,

$$\int_s^\infty (6u_1 + 2v_2 + 2v_1v - 2u_2u - u_1^2 - 2w)(x) dx = 2w_1 - 3u_2 + u_1v_0 - u_0\tilde{v}_1 \quad (3.20)$$

We find that the large n -expansion of the probability distribution function of the largest eigenvalue for the GUE_n case is given by the following; if we set

$$\tau(s) = \sqrt{2(n+c)} + \frac{s}{2^{\frac{1}{2}}n^{\frac{1}{6}}}, \quad (3.21)$$

then as $n \rightarrow \infty$, we have

$$\begin{aligned} F_{n,2}(\tau(s)) &= F_2(s) \left\{ 1 + cu(s)n^{-\frac{1}{3}} + \right. \\ &\left. \frac{n^{-\frac{1}{3}}}{20} \left[(20c^2 - 3)v(s) + \int_s^\infty (-6u_1 - 2v_2 - 2v_1v + 2u_2u + u_1^2 + 2w)(x) dx \right] \right\} + O(n^{-1}) \end{aligned} \quad (3.22)$$

uniformly in s .

4 Conclusion

Our motivation in this paper was to find large n -expansion of q_n and p_n . The importance of such large n expansions is that they not only give a direct proof of Theorem 1.3 (GUE_n case), but they are essential ingredients in extending Theorem 1.3 to the GOE_n and GSE_n cases. We will return to these cases in a subsequent paper.

³The derivation was simple for the term containing the constant c since we can trace it out.

Acknowledgements: The author would like to thank Professor Craig Tracy for the discussions that initiated this work and for the invaluable guidance, and the Department of Mathematical Sciences at the University of Alabama in Huntsville.

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