Midterm #2

Full Name: .... **Signature**..

Note: You need to give **Justification** to all your **Answers** in order to have full **CREDIT**.

Exercise 1. *(40 points)*

Is the following statements True or False? Justify your answers .

1. $\mathbb{Z}_4 \diagdown \{[0]\}$ *is a group with respect to* $\odot$

2. (S_7, o) *is commutative.*

3. *If* $(ab, c) = 1$ *then* $(a, c) = 1$ *and* $(b, c) = 1$.

4. *If* c/ab *and* $(a, c) = d$ *then* c/bd.

5. *The number of positive divisors of* $2^r 3^s$ *equals the number of positive divisors of* $5^s 7^r$.

6. *If* $(G, \cdot)$ *is a group and* $\exists\, a, b \in G$ *such that* $ab = ba$ *then* G *is commutative.*

7. *Let* a, b *be two elements of a group* G. *Then* $ab = ba$ *iff* $(ab)^2 = a^2 b^2$.

8. $\mathbb{Z}_5 \times \mathbb{Z}_{11}$ *is a cyclic group.*

9. *In* S_3, $[S_3; H] = 2$ *if* $H = \{(1), (12)\}$

10. *If* G *is a group and* $|G| = p$ *with* p *prime, then* G *has* $p + 1$ *subgroups.*

Exercise 2. *(30 points) Let $\Phi : G \longrightarrow H$ be a group Homomorphism*

1. *Let K be a subgroup of H. Prove that $\Phi^{-1}(K) = \{g \in G; \Phi(g) \in K\}$ is a subgroup of G.*

2. *Let A be a subgroup of G. Prove that $\Phi(A)$ is a subgroup of H*

3. *Let $\Phi_a : \mathbb{Z} \longrightarrow \mathbb{Z}_8$ be defined by $\Phi_a(k) = [ak]_8$. Φ_a is a group homomorphism for each positive integer a.*

   (a) *$K = \{[0], [2], [4], [6]\}$ is a subgroup of $\mathbb{Z}_8$, find $\Phi_2^{-1}(K)$*

   (b) *$A = 3\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, find $\Phi_2(A)$*

Exercise 3. *(20 points)*

1. *Let H be a subgroup of a group G such that $|H| = 8$, $[G : H] > 3$ and $|G| < 41$. What are the possibilities for $|G|$?*

2. *Prove that if G is a group of order p^2 with p prime and G not cyclic, then $a^p = e$ for each $a \in G$.*

Exercise 4. *(20 points)*
Let G be a group and A be a nonempty subset of G. The Centralizer of A in G is the set $C_G(A) = \{g \in G| \quad gag^{-1} = a, \quad \forall a \in A\}$. This is the set of elements of G which commute with every element of A.

1. *Show that $C_G(A)$ is a subgroup of G.*

2. (a) *Find the center of G defined by $Z(G) = C_G(G)$ if G is Abelian.*

   (b) *Find $C_G(A)$ if $A = \{(12), (1)\}$ and $G = S_3$*