JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **165**, 71–101 (1992)

Asymptotic Integration of Delay Differential Systems

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1. INTRODUCTION

The purpose of this paper is to completely prove a conjecture in J. R. Haddock and R. Sacker [1] and further extend a result on asymptotic integration obtained previously by O. Arino and I. Gyori [2–4].

Asymptotic integration deals with non-autonomous evolution equations which asymptotically are autonomous, and aims at relating the asymptotic behavior of the solutions of these equations to the asymptotic behavior of the solutions of the limit equation. Classical results on this problem exist for ordinary differential equations (i.e., cf. [5–9]). For delay differential equations, the earliest results are due to K. L. Cooke [10], and some later results can be found in [1–4, 11–16].

In [1], in search of an extension of results by Hartman [5], Hartman and Wintner [6], Atkinson [7], and Harris and Lutz [8], notably for ordinary differential equations of the form

$$\dot{x}(t) = (\Lambda + A(t)) \cdot x(t), \tag{1.1}$$

where $\Lambda = \operatorname{diag}\{\lambda_1, ..., \lambda_n\}$ is a real diagonal matrix, $\lambda_i \neq \lambda_j$, $i \neq j$, and $A(\cdot)$ is a continuous matrix function defined on R^+, $R^+ = [0, +\infty)$, and L^2 perturbation, Haddock and Sacker conjectured an asymptotic formula for the solutions of the delay equation

$$\dot{x}(t) = (\Lambda + A(t))\, x(t) + B(t)\, x(t-r), \tag{1.2}$$

where $r > 0$ is a constant, Λ and A are the same as in (1.1), and $B(\cdot)$ is a matrix function and is in L^2. They stated that there exists a matrix function $F(\cdot)$, $F(t) \to 0$ as $t \to +\infty$, such that for each solution x of (1.2), there exist a constant vector c and a function $f(\cdot)$, $f(t) \to 0$ as $t \to \infty$, such that

$$x(t) = (Id + F(t)) \cdot \exp\left(\int_0^t \Lambda(s)\, ds\right) \cdot (c + f(t)), \tag{1.3}$$

0022-247X/92 \$3.00

where

$$\Lambda(t) = \Lambda + \operatorname{diag}\{A(t)\} + \operatorname{diag}\{B(t)\} \cdot e^{-r\Lambda}.$$

The conjecture was proved in the scalar case in [1], and further proved in the case of "quasi-triangular" systems by Arino and Gyori [2–4]. In these two situations, the obtained results show that the formula (1.3) holds with $F(t) = 0$. For the general situation, a weaker result was also obtained in [2–4]. Indeed, Arino and Gyori in [2–4] considered a general system

$$\dot{x}(t) = \Lambda \cdot x(t) + L(t, x_t), \tag{1.4}$$

where Λ is the same as above, and $L\colon R^+ \times C([-r, 0], E^n) \to E^n$ is continuous, with $L(t, \cdot)$ linear on $C([-r, 0], E^n)$ and $\|L(t, \cdot)\|$, the norm of $L(t, \cdot)$, in L^2, and $x(t) \in E^n$, x_t denotes, as usual, the function defined on $[-r, 0]$ by $x_t(s) = x(t+s)$, $-r \leqslant s \leqslant 0$, here r is the maximum delay in (1.4). By an inductive method, they obtained the following result: there exists a functional $G(t)$ defined on the space $C([-2r, t], E^n)$, $\|G(t)\| \to 0$ as $t \to +\infty$, such that for each solution x of (1.4), there exist a constant vector c, a function η_1 with values in E^n, $\eta_1(t) \to 0$ as $t \to +\infty$, and a function η_2, $\eta_2(t) \in C([-2r, t], E^n)$, $\eta_2(t) \to 0$ as $t \to +\infty$ such that

$$\begin{aligned} x(t) = \exp\left\{\int_0^t \Lambda(s)\, ds\right\} \cdot [c + \eta_t(t)] \\ + G(t) \cdot \left\{\exp \cdot \int_0^t \Lambda(s)\, ds \cdot [c + \eta_2(t)]\right\}, \end{aligned} \tag{1.5}$$

where $\Lambda(t) = \Lambda + \operatorname{diag}\{L(t, \exp(\Lambda \cdot))\}$. Obviously, when applying this result to (1.2), the obtained formula for the solutions of (1.2) by (1.5) is generally less agreeable than (1.3) for in (1.3) the function $F(t)$ is replaced by a functional $G(t)$ defined on the space $C([-2r, t], E^n)$. So, the conjecture still remains to be proved.

In this paper, we consider a more general system

$$\dot{x}(t) = (\Lambda + V(t)) \cdot x(t) + L(t, x_t), \tag{1.6}$$

where $x(t) \in E^n$, E^n is the n-dimensional real Euclidean space, with the norm $|x| = \sum_{i=1}^n |x_i|$ for $x = \operatorname{col}(x_1, ..., x_n)$ in E^n, and $E = E^1$. We will state now the assumptions on Λ, V, and L.

(H_1) $\Lambda = \operatorname{diag}\{\lambda_1, ..., \lambda_n\}$, with $\lambda_i \neq \lambda_j$, $i \neq j$;

(H_2) $V(t) = \operatorname{diag}\{v_1(t), ..., v_n(t)\}$, with $v_i\colon R^+ \to E$ continuous and $\sup_{s \geqslant t} (1 + s - t)^{-1} \cdot \int_t^s v_i(r)\, dr \to 0$ as $t \to \infty$, $i = 1, 2, ..., n$;

(H_3) For each $t \in R^+$, $L(t, \cdot)$ is linear continuous from $C([-r, 0], E^n)$ into E^n, $t \to L(t, \cdot)$ is continuous, and $\|L(t, \cdot)\|$ is in $L^p(R^+)$, $1 \leqslant p \leqslant 2$. Here $r > 0$ is the maximum delay in (1.6), and

$C[-r, 0], E^n)$ is the Banach space of continuous functions mapping the interval $[-r, 0]$ into E^n with the norm $\|\varphi\| = \sup_{-r \leqslant s \leqslant 0} |\varphi(s)|$. Clearly, the situation under consideration here covers the those in [1–4]. By a different method from the one used in [2–4], we get an asymptotic formula for the solutions of (1.6). When applying our result to (1.2), we obtain a slightly stronger result than the above conjecture. Also, our result is stronger than the one in [2–4], at least in the general case (still, Arino and Gyori's result for quasi-triangular systems is stronger).

The method used in this paper is a similar to the one employed in [11]. That is, we first get a class of "special solutions" of (1.6), and then using these solutions we obtain the asymptotic formula for all solutions of (1.6).

The plan of our paper is as follows. In Section 2, we establish some lemmas needed in later discussion. In Section 3, using Lemma 4 established in Section 2 and a fixed point theorem we get a class of "special solutions" of (1.6). In the last section, using these special solutions we obtain an asymptotic formula for all solutions of (1.6).

2. Some Lemmas

Lemma 1. *Let $\psi \in L^p$, $1 \leqslant p \leqslant 2$, be nonnegative and continuous on R^+. For $t \geqslant 0$, $\varepsilon > 0$, define*

$$\sigma(t, \varepsilon) = \int_0^t \psi(s)\, e^{-\varepsilon(t-s)}\, ds, \qquad \zeta(t, \varepsilon) = \int_t^\infty \psi(s)\, e^{-\varepsilon(s-t)}\, ds. \tag{2.1}$$

Then, for any $\varepsilon > 0$,

(1) $\sigma(t, \varepsilon) \to 0$, $\zeta(t, \varepsilon) \to 0$, $t \to \infty$;

(2) $\sigma(\cdot, \varepsilon) \in L^q$, $\zeta(\cdot, \varepsilon) \in L^q$, $1/p + 1/q = 1$.

Proof. The first part of Lemma 1 concluded from [5, Example 4.1, p. 286]. From [5, Example 4.2, p. 286], we see that $\sigma(\cdot, \varepsilon) \in L^p$, and $\zeta(\cdot, \varepsilon) \in L^p$. Noticing that $p \leqslant 2 \leqslant q$ and the result (1), we can get the second part of Lemma 1.

Lemma 2. *Let ψ be the same as in Lemma 1. Let $v: R^+ \to E$ be continuous, with $\sup_{s \geqslant t} (1 + s - t)^{-1} \cdot \int_t^s v(r)\, dr \to 0$ as $t \to \infty$. For $\alpha > 0$, $t_0 \geqslant 0$, define $\sigma_1(\cdot, \alpha, t_0)$, $\zeta_1(\cdot, \alpha, t_0)$ on R^+ as*

$$\sigma_1(t, \alpha, t_0) = \int_{t_0}^t \psi(s) \cdot \exp\left(-\int_s^t \lambda(r)\, dr\right) ds,$$

$$\zeta_1(t, \alpha, t_0) = \int_t^\infty \psi(s) \cdot \exp\left(-\int_t^s \lambda(r)\, dr\right) ds,$$

where $\lambda(t)=\alpha+v(t)$, *for* $t\geqslant 0$. *Then, for any constant* ε, *such that* $0<\varepsilon<\alpha$, *there exists a constant* $t_0\geqslant 0$, *such that for* $t\geqslant t_0$,

$$\sigma_1(t,\alpha,t_0)\leqslant e^{\alpha-\varepsilon}\cdot\sigma(t,\varepsilon),\qquad \zeta_1(t,\alpha,t_0)\leqslant e^{\alpha-\varepsilon}\cdot\zeta(t,\varepsilon),$$

where $\sigma(t,\varepsilon)$, $\zeta(t,\varepsilon)$ *are defined as in Lemma* 1.

Proof. Since $\sup_{s\geqslant t}(1+s-t)^{-1}\cdot\int_t^s v(r)\,dr\to 0$ as $t\to\infty$, for any given number ε, $0<\varepsilon<\alpha$, there exists a number $t_0\geqslant 0$, such that for $s\geqslant t\geqslant t_0$,

$$(1+s-t)^{-1}\left|\int_t^s v(r)\,dr\right|<\alpha-\varepsilon,$$

i.e.,

$$\left|\int_t^s v(r)\,dr\right|<(\alpha-\varepsilon)\cdot(1+s-t).$$

So, for $t\geqslant s\geqslant t_0$,

$$\begin{aligned}\exp\left(-\int_s^t \lambda(r)\,dr\right)&=\exp\left(-\alpha(t-s)-\int_s^t v(r)\,dr\right)\\&\leqslant \exp(-\alpha(t-s)+(\alpha-\varepsilon)\cdot(t-s+1))\\&\leqslant e^{\alpha-\varepsilon}\cdot e^{-\varepsilon(t-s)},\end{aligned}$$

and so, for $t\geqslant t_0$,

$$\sigma_1(t,\alpha,t_0)\leqslant e^{\alpha-\varepsilon}\int_{t_0}^t \psi(s)\,e^{-\varepsilon(t-s)}\,ds\leqslant e^{\alpha-\varepsilon}\cdot\alpha(t,\varepsilon).$$

This completes the first inequality of Lemma 2. The second one can be proved in a similar way.

LEMMA 3. *Let* $\varphi\in L^p$, $1\leqslant p\leqslant 2$, *be continuous with* $\varphi(t)=\varphi(0)$ *for* $-r\leqslant t\leqslant 0$. *For* $t\geqslant 0$, *let*

$$h(t,\theta)=\left(\exp\int_{t+\theta}^t \varphi(s)\,ds\right)-1,\qquad -r\leqslant\theta\leqslant 0.$$

Then,

(1) $\|h(t,\cdot)\|^p=O(\int_{t-r}^t|\varphi(t)|^p\,ds)$ *as* $t\to\infty$,

(2) $\int_0^\infty\|h(t,\cdot)\|^q\,dt<\infty$, $1/p+1/q=1$,

where $\|h(t,\cdot)\|=\sup_{-r\leqslant\theta\leqslant 0}|h(t,\theta)|$.

Proof. Since $\varphi \in L^p$, we see that $|\int_t^{t+\theta} \varphi(s)\,ds| \to 0$ as $t \to \infty$ uniformly in θ, $-r \leqslant \theta \leqslant 0$. By the Holder inequality, we deduce

$$\begin{aligned}\|h(t,\cdot)\| &= \sup_{-r\leqslant\theta\leqslant 0} |h(t,\theta)| = \sup_{-r\leqslant\theta\leqslant 0} O\left(\left|\int_{t+\theta}^{t} \varphi(s)\,ds\right|\right)\\ &= O\left(\int_{t-r}^{t} |\varphi(s)|\,ds\right)\\ &= O\left(\left(\int_{t-r}^{t} |\varphi(s)|^p\,ds\right)^{1/p}\right) \qquad \text{as} \quad t \to \infty,\end{aligned}$$

and so,

$$\|h(t,\cdot)\|^p = O\left(\int_{t-r}^{t} |\varphi(s)|^p\,ds\right) \qquad \text{as} \quad t \to \infty.$$

Thus, we complete the first part of this lemma.

From (1), it follows that $\int_0^\infty \|h(t,\cdot)\|^p\,dt < \infty$, and $\|h(t,\cdot)\| \to 0$ as $t \to \infty$. Noting that $q \geqslant 2 \geqslant p$, we can immediately get the second part of this lemma.

LEMMA 4. *Consider the equation*

$$\begin{aligned}\dot{x}(t) &= M(t, x_t) + N(t, y_t) + R_1(t),\\ \dot{y}(t) &= P(t, x_t) + Q(t, y_t) + R_2(t),\end{aligned} \tag{2.2}$$

with $x \in E^n$, $y \in E^m$,

$$\begin{aligned}&M(resp.\ P)\colon [t_0, +\infty) \times C([-r, 0], E^n) \to E^n(resp.\ E^m),\\ &N(resp.\ Q)\colon [t_0, +\infty) \times C([-r, 0], E^m) \to E^n(resp.\ E^m),\\ &R_1(resp.\ R_2)\colon [t_0, +\infty) \to E^n(resp.\ E^m),\end{aligned}$$

where M, N, P, Q *are continuous linear functionals with respect to the second variable. Assume that*

(H_4) *the equation*

$$\dot{x}(t) = M(t, x_t) \tag{2.3}$$

is stable;

(H_5) *the equation*

$$\dot{y}(t) = Q(t, y_t) \tag{2.4}$$

is exponentially stable;

(H_6) $\|N(t, \cdot)\|$ *and* $\|P(t, \cdot)\|$ *are in* L^p, *with* $1 \leqslant p \leqslant 2$;

(H_7) $R_1(\cdot)$ *and* $R_2(\cdot)$ *are in* L^1.

Let (x, y) *be a solution of* (2.2). *Then* x *is bounded and* $\lim_{t \to +\infty} y(t) = 0$. *Moreover, if for all the solutions* $u(t)$ *of* (2.3), $\lim_{t \to +\infty} u(t)$ *exists, the same holds with the solutions of* (2.2).

The proof of Lemma 4 can be finished in a very similar way to the one of Proposition 2 in [4], so we omit it. In fact, if $R_1(t) = R_2(t) = 0$, this lemma becomes Proposition 2 of [4]. From the proof, we can also get the following important estimates, which are very useful in proving our Theorem 1, for the solutions of (2.2) for $t \geqslant t_0$ (where t_0 is large enough, such that $c(t_0) < 1$, $c(t_0)$ is given below):

$$\|x_t(t_0, x_{t_0}, y_{t_0})\| \leqslant (1 - c(t_0))^{-1} \cdot b(t_0, \|x_{t_0}\|, \|y_{t_0}\|), \tag{2.5}$$

$$\begin{aligned} \|y_t(t_0, x_{t_0}, y_{t_0})\| \leqslant{}& Ke^{-\alpha(t-t_0)} \|y_{t_0}\| + K \int_{t_0}^{t} e^{-\alpha(t-s)} |R_2(s)| \, ds \\ &+ K \int_{t_0}^{t} e^{-\alpha(t-s)} p(s) \, |x_s| \, ds, \end{aligned} \tag{2.6}$$

where K, α are positive numbers, only dependent on Eqs. (2.3) and (2.4), $n(\cdot)$ and $p(\cdot)$ are in L^p, such that

$$|N(t, \varphi)| \leqslant n(t) \cdot \|\varphi\|, \qquad |P(t, \psi)| \leqslant p(t) \, \|\psi\|,$$

$$\begin{aligned} b(t_0, u_0, v_0) ={}& Ku_0 + K^2 \left(\int_{t_0}^{\infty} n(s) \, e^{-\alpha(s-t_0)} \, ds \right) \\ &\cdot v_0 + K \int_{t_0}^{\infty} |R_1(s)| \, ds \\ &+ K^2 \int_{t_0}^{\infty} n(s) \left(\int_{t_0}^{s} e^{-\alpha(s-r)} \, |R_2(r)| \, dr \right) ds, \end{aligned} \tag{2.7}$$

$$c(t_0) = K^2 \int_{t_0}^{\infty} n(s) \left(\int_{t_0}^{s} e^{-\alpha(s-r)} \cdot p(r) \, dr \right) ds. \tag{2.8}$$

It is not difficult to see that b and c are uniformly bounded with respect to t_0 and moreover $c(t_0) \to 0$ as $t_0 \to +\infty$.

Corresponding to Corollary 1 in [4], we have the following corollary.

COROLLARY 1. *Asssume* (H_5), (H_6), (H_7), *and that* $\|M(t, \cdot)\|$ *is in* L^1.

Let (x, y) *be a solution of* (2.2). *Then* $\lim_{t\to+\infty} y(t)=0, \lim_{t\to+\infty} x(t)$ *exists, and have*

$$\lim_{t\to+\infty} x(t)=x(t_0)+w(t_0, x_{t_0}, y_{t_0}, R_1, R_2)+\int_{t_0}^{\infty} R_1(s)\,ds, \tag{2.9}$$

where w *is a mapping from* $R^+\times C([-r,0],E^n)\times C([-r,0],E^m)\times L^l(R^+,E^n)\times L^1(R^+,E^m)\to E^n$ *and have the property that for each* $t_0, R_1, R_2, w(t_0,\cdot,\cdot,R_1,R_2)$ *satisfies the Lipschitz condition with respect to the second and third variables. That is, there exists a function* $d(\cdot)\colon R^+\to R^+, d(t_0)\to 0$ *as* $t_0\to+\infty$, *and independent of* R_1 *and* R_2, *such that*

$$|w(t_0,\varphi_1,\psi_1,R_1,R_2)-w(t_0,\varphi_2,\psi_2,R_1,R_2)| \leqslant d(t_0)\cdot(\|\varphi_1-\varphi_2\|+\|\psi_1-\psi_2\|). \tag{2.10}$$

Moreover, there exists a constant $T\geqslant 0$, *which only depends on* M, N, P, *and* Q, *such that if* $t_0\geqslant T$, *then for each* c *in* E^n, *there exists a solution of* (2.2) *defined on* $[t_0-r,+\infty)$ *such that* $\lim_{t\to+\infty} x(t)=c$.

Proof. From Proposition 1 in [4], we deduce that Eq. (2.3) is stable and its solutions converge. So, all the conditions of Lemma 4 are satisfied, which yields that $\lim_{t\to+\infty} y(t)=0$, $\lim_{t\to+\infty} x(t)$ exists for each solution (x, y) of (2.2). We now show (2.9). Let (x, y) be a solution of (2.2). By the bound of x and $\|M(t,\cdot)\|$ is in L^1, it follows that $M(t, x_t)$ is in L^1. By the estimate (2.6) and Lemma 1, we see that y_t is in L^q, and so $N(t, y_t)$ is in L^1. So, since $R_1(\cdot)$ is in L^1, by the first equation of (2.2) we get

$$\lim_{t\to+\infty} x(t_0, x_{t_0}, y_{t_0})=x(t_0)+\int_{t_0}^{\infty} M(s, x_s)\,ds + \int_{t_0}^{\infty} N(s, y_s)\,ds+\int_{t_0}^{\infty} R_1(s)\,ds,$$

and comparing this expression to (2.9), we have that

$$w(t_0, x_{t_0}, y_{t_0}, R_1, R_2)=\int_{t_0}^{\infty} M(s, x_s)\,ds+\int_{t_0}^{\infty} N(s, y_s)\,ds. \tag{2.11}$$

Now, for φ_1, φ_2 in $C([-r,0],E^n)$ and ψ_1, ψ_2 in $C([-r,0],E^m)$, let $(x_i, y_i)=(x(t_0,\varphi_i,\psi_i), y(t_0,\varphi_i,\psi_i))$, $i=1, 2$. Let $m\in L^l$ such that $\|M(t,\cdot)\|\leqslant m(t)$. Then, by (2.11)

$$w(t_0, \varphi_1, \psi_1, R_1, R_2) - w(t_0, \varphi_2, \psi_2, R_1, R_2)$$
$$= \int_{t_0}^{\infty} M(s, x_{1s} - x_{2s})\, ds + \int_{t_0}^{\infty} N(s, y_{1s} - y_{2s})\, ds$$
$$\leqslant \int_{t_0}^{\infty} m(s)\, \|x_{1s} - x_{2s}\|\, ds + \int_{t_0}^{\infty} n(s)\, \|y_{1s} - y_{2s}\|\, ds. \tag{2.12}$$

Noticing that $(x_1 - x_2, y_1 - y_2)$ is a solution of (2.2) in the case $R_1(t) = R_2(t) = 0$, so by (2.5) and (2.6) we have

$$\text{the left of (2.12)}$$
$$\leqslant (1 - c(t_0))^{-1} \cdot b(t_0, \|\varphi_1 - \varphi_2\|, \|\psi_1 - \psi_2\|)$$
$$\cdot \left(\int_{t_0}^{\infty} m(s)\, ds + K \int_{t_0}^{\infty} n(s) \left(\int_{t_0}^{\infty} e^{-\alpha(s-r)} p(r)\, dr \right) ds \right)$$
$$+ K \int_{t_0}^{\infty} n(s)\, e^{-\alpha(s-t_0)}\, ds\, \|\psi_1 - \psi_2\|. \tag{2.13}$$

Now let

$$d(t_0) = (1 - c(t_0))^{-1} \left(K + K^2 \left(\int_{t_0}^{\infty} n(s)\, e^{-\alpha(s-t_0)}\, ds \right) \right)$$
$$\cdot \left(\int_{t_0}^{\infty} m(s)\, ds + K \int_{t_0}^{\infty} n(s) \left(\int_{t_0}^{\infty} e^{-\alpha(s-r)} p(r)\, dr \right) ds \right)$$
$$+ K \int_{t_0}^{\infty} n(s)\, e^{-\alpha(s-t_0)}\, ds. \tag{2.14}$$

By (2.7), (2.13), (2.14), we can easily deduce (2.10). Clearly, from (2.14) we see that $d(t_0)$ depends only on M, N, P, Q, no relation with R_1 and R_2, and that by Lemma 1, $d(t_0) = o(1)$ as $t_0 \to +\infty$. Thus, the first part of Corollary 1 is proved.

In order to show the second part, let us restrict our attention to constant date $x_{t_0} = x_0$ and $y_{t_0} = 0$. From (2.9) we get

$$\lim_{t \to +\infty} x(t_0, x_0, 0)(t) = x_0 + w_1(t_0, x_0, R_1, R_2)$$
$$+ \int_{t_0}^{\infty} R_1(s)\, ds, \tag{2.15}$$

where $w_1 : R^+ \times E^n \times L^1(R^+, E^n) \times L^1(R^+, E^m) \to E^n$ is defined by $w_1(t_0, x_0, R_1, R_2) = w(t_0, x_0, 0, R_1, R_2)$. Now let $0 < a < 1$ be an arbitrary constant. For such a fixed constant a, since $d(t_0) = o(1)$ as $t_0 \to +\infty$ we see that there exists a number $T \geqslant 0$, which is independent of R_1 and R_2, such that for any $t_0 \geqslant T$, $d(t_0) \leqslant a$, and so by (2.10) we have

$$|w_1(t_0, x_0, R_1, R_2) - w_1(t_0, \bar{x}_0, R_1, R_2)$$
$$\leqslant a\,|x_0 - \bar{x}_0|, \qquad \text{for} \quad x_0, \bar{x}_0 \text{ in } E^n. \tag{2.16}$$

So, by a well-known result we see that the mapping defined by the right side of (2.15) from E^n into E^n is surjective (in fact is an isomorphism). This gives the desired result of the second part.

Remark 2.1. By the second part of Corollary 1, we see that for all $(R_1, R_2) \in L^1(R^+, E^n) \times L^1(R^+, E^m)$ there exists a common $T \geqslant 0$, which depends only on M, N, P, Q, and have the property as stated as the second part of Corollary 1.

LEMMA 5. *Assume* (H_5), (H_6), (H_7) *and that* $\|M(t, \cdot)\|$ *is in* L^p, *with* $1 \leqslant p \leqslant 2$. *Then, there exist a constant* $T_1 \geqslant 0$ *and two bounded functions* $K_1(\cdot)$ *and* $K_2(\cdot)$ *defined on* $[T_1, +\infty)$, *which are all determined by* M, N, P, *and* Q, *such that for each solution* $(x, y) = (x(t_0, x_{t_0}, y_{t_0}), y(t_0, x_{t_0}, y_{t_0}))$ *of* (2.2) *with* $t_0 \geqslant T_1$, *the following estimates hold for* $t_0 \leqslant t \leqslant t_0 + r$:

$$\|x_t\| \leqslant K_1(t_0) \cdot (\|x_{t_0}\| + \|y_{t_0}\|)$$
$$+ (1 - c_1(t_0))^{-1} \cdot b_1(t_0, R_1, R_2), \tag{2.17}$$
$$\|y_t\| \leqslant K_2(t_0) \cdot (\|x_{t_0}\| + \|y_{t_0}\|) + b_2(t_0, R_1, R_2),$$

where c_1, b_1, *and* b_2 *are given by*

$$c_1(t_0) = \int_{t_0}^{t_0 + r} m(s)\, ds + K \int_{t_0}^{t_0 + r} n(s) \cdot \left(\int_{t_0}^{s} e^{-\alpha(s - \tau)} p(\tau)\, d\tau \right) ds,$$

$$b_1(t_0, R_1, R_2) = \int_{t_0}^{t_0 + r} |R_1(s)|\, ds + K \int_{t_0}^{t_0 + r} n(s)$$
$$\cdot \left(\int_{t_0}^{s} e^{-\alpha(s - \zeta)}\, |R_2(\zeta)|\, d\zeta \right) ds,$$

$$b_2(t_0, R_1, R_2) = K(1 - c_1(t_0))^{-1} \cdot b_1(t_0, R_1, R_2) \cdot \int_{t_0}^{t_0 + r} p(s)\, ds$$
$$+ K \int_{t_0}^{t_0 + r} |R_2(s)|\, ds, \tag{2.18}$$

in which K *and* α *are the same constants as in Lemma* 4, *and* $m(\cdot)$ *is in* L^p *such that* $|M(t, \varphi)| \leqslant m(t) \cdot \|\varphi\|$ *for* $\varphi \in C([-r, 0], E^n)$.

Proof. Let (x, y) be a solution of (2.2). From the first equation of (2.2) we can easily get

$$\|x_t\| \leqslant \|x_{t_0}\| + \int_{t_0}^{t} m(s)\,\|x_s\|\,ds + \int_{t_0}^{t} n(s)\,\|y_s\|\,ds + \int_{t_0}^{t} |R_1(s)|\,ds. \tag{2.19}$$

Noticing that in this Lemma, (2.6) still holds so replacing $\|y_s\|$ in (2.19) by the right side of (2.6) we can deduce (2.17), in which c_1 and b_1 are given in (2.18), $T_1 \geqslant 0$ is such a number that for $t_0 \geqslant T_1$, $c_1(t_0) < 1$ (such a number exists since $c_1(t_0) = o(1)$ as $t_0 \to +\infty$), and $K_1(t_0)$ can be chosen as $K_1(t_0) = (1 - c_1(t_0))^{-1} \cdot (1 + K \int_{t_0}^{t_0+r} n(s)\, e^{-\alpha(s-t_0)}\, ds)$. Now, using the obtained estimate for $\|x_t\|$ to the right side of (2.6), we can get the second one of (2.17), where $b_2(t_0, R_1, R_2)$ is given in (2.18), and $K_2(t_0)$ is chosen as

$$K_2(t_0) = K(1 + K_1(t_0) \cdot \int_{t_0}^{t_0+r} p(s)\, ds).$$

By the choice of T_1 and expressions of $K_1(\cdot)$ and $K_2(\cdot)$, it is clear that they are all determined completely by M, N, P, and Q, and are no relation with $R_1(\cdot)$ and $R_2(\cdot)$. This completes the proof of Lemma 5.

Lemma 6. *Let ρ be measurable, locally bounded, and nonnegative on $[t_0 - r, +\infty)$, where $r > 0$ and $t_0 \geqslant 0$ are constants, and m be nonnegative on $[t_0, +\infty)$ and in L^p. Suppose that*

$$\rho(t) \leqslant m(t) \cdot \int_{t-r}^{t} \rho(s)\, ds, \qquad \textit{for} \quad t \geqslant t_0. \tag{2.20}$$

Then,

(1) *for any $\alpha < 0$, $\int_{t_0}^{\infty} e^{\alpha t} \rho(t)\, dt < +\infty$;*

(2) *for any $\alpha > 0$, $\int_{t}^{\infty} \rho(s)\, ds = o(e^{-\alpha t})$ as $t \to +\infty$.*

Remark 2.2. This lemma is a special case of Corollary 1 in [17]. For the sake of clarity we give a direct proof.

Proof. Since m is in L^p, we see that $\int_t^{t+r} m(s)\, ds = o(1)$ as $t \to +\infty$. So, there exists a $\bar{t}_0 \geqslant t_0 + r$, such that for $t \geqslant \bar{t}_0$, $e^{\alpha r} \int_t^{t+r} m(s)\, ds \leqslant 1/2$. Now, multiplying both sides of (2.20) by $e^{\alpha t}$ and integrating from $\bar{t}_0$ to T $(T \geqslant \bar{t}_0)$ we obtain

$$\int_{\bar{t}_0}^{T} e^{\alpha t}\rho(t)\,dt \leqslant \int_{\bar{t}_0}^{T} e^{\alpha t} m(t)\cdot\int_{t-r}^{t}\rho(s)\,ds\,dt$$

$$\leqslant c + \int_{\bar{t}_0}^{T}\rho(s)\cdot\int_{s}^{s+r} e^{\alpha t} m(t)\,dt\,ds, \tag{2.21}$$

where $c = \int_{\bar{t}_0 - r}^{\bar{t}_0}\rho(s)\cdot\int_s^{s+r} e^{\alpha t} m(t)\,dt\,ds$. Changing the variable of integration in the right-hand side of (2.21) to t and moving it to the left-hand side of (2.21) we get

$$(1/2)\int_{\bar{t}_0}^{T} e^{\alpha t}\cdot\rho(t)\,dt \leqslant \int_{\bar{t}_0}^{T}\left(1 - e^{\alpha r}\int_{t}^{t+r} m(s)\,ds\right)\cdot e^{\alpha t}\cdot\rho(t)\,dt$$

$$\leqslant \int_{\bar{t}_0}^{T}\left(e^{\alpha t} - \int_{t}^{t+r} e^{\alpha s} m(s)\,ds\right)\cdot\rho(t)\,dt \leqslant c,$$

and so

$$\int_{\bar{t}_0}^{T} e^{\alpha t}\cdot\rho(t)\,dt \leqslant 2c. \tag{2.22}$$

This implies (1). From (1) we easily conclude (2). Thus, this lemma is proved.

3. Special Solutions

In this section, we shall get a class of solutions of (1.6) with special asymptotic behavior more or less like $\exp(\lambda_i t)\,v$, $i = 1, 2, ..., n$, which will be called the special solutions of (1.6).

We first define for each integer k, $1 \leqslant k \leqslant n$,

$$\lambda_k(t) = \lambda_k + v_k(t), \text{ for } t \geqslant 0, \qquad \text{and} \qquad \lambda_k(t) = \lambda_k(0), \text{ for } -r \leqslant t \leqslant 0,$$

$$w_k(t, t_0) = \int_{t_0}^{t}\lambda_k(s)\,ds, \qquad W(t, t_0) = \operatorname{diag}\{w_1(t, t_0), ..., w_n(t, t_0)\},$$

$$\delta_k(t) = e_k^T\cdot L(t, \exp[w_k(t + \cdot, t)]\,e_k), \qquad s_k(t, t_0) = \int_{t_0}^{t}\delta_k(s)\,ds,$$

$$S(t, t_0) = \operatorname{diag}\{s_1(t, t_0), ..., s_n(t, t_0)\}, \qquad t \geqslant -r,\ t_0 \geqslant 0, \tag{3.1}$$

where $e_k^T = (0, ..., 0, 1, 0, ..., 0)$, and then introduce an operator P_k mapping from $C([-r,0], E^n)$ to $C([-r,0]\, E^k)$, such that $P_k\varphi = \mathrm{col}(\varphi_1, ..., \varphi_k)$ for $\varphi = \mathrm{col}(\varphi_1, ..., \varphi_n)$ in $C([-r,0], E^n)$. Now we shall state our first theorem under the assumption that the λ_i's are ordered: for $i<j$, $\lambda_i<\lambda_j$, since this assumption does not restrict the generality.

THEOREM 1. *For each integer k, $1 \leqslant k \leqslant n$, there exists a number $t_k \geqslant 0$, such that for any $t_0 \geqslant t_k$ and each φ^k in $C([-r,0], E^k)$, there exists at least one solution $x(t_0, \varphi^k)$ of* (1.6), *which is defined on* $[t_0 - r, +\infty)$ *and satisfies $P_k x_{t_0}(t_0, \varphi^k) = \varphi^k$, such that as $t \to +\infty$*

$$x(t_0, \varphi^k)(t) = \exp\{w_k(t, t_0) + s_k(t, t_0)\} \cdot (be_k + o(1)), \tag{3.2}$$

where w_k and s_k are defined by (3.1), *and b is a constant dependent on $x(t_0, \varphi^k)$, and e_k is a kth unit vector in E^n. Moreover, there exists a number $\bar{t}_k \geqslant 0$, such that for any $t_0 \geqslant \bar{t}_k$ and any given constant b, there exists a solution $x(t_0, k, b)$ of* (1.6), *which is defined on $[t_0 - r, +\infty)$, such that $x(t_0, k, b)(t)$ is equal to the right-hand side of* (3.2) *as $t \to +\infty$. Such solutions $x(t_0, \varphi^k)$ and $x(t_0, k, b)$, $1 \leqslant k \leqslant n$, are called special solutions of* (1.6).

Remark 3.1. When applying this theorem to Eq. (1.1), we can obtain a stronger result than the one obtained in [6, p. 71, Result (i)]. For in [6], the result is only the second part of our theorem.

The proof given below is based on a change of variable and then on employing the alternative method to the transformed equation. In the process of the proof, we shall use Corollary 1 and Lemma 5 and the estimates (2.5), (2.6), and (2.17).

Proof. The first step of the proof is to make the change of variable

$$x(t) = \exp\{w_k(t, t_0) + s_k(t, t_0)\} \cdot y(t), \qquad t_0 \geqslant 0, t \geqslant t_0 - r, \tag{3.3}$$

Then, for $t \geqslant t_0$ Eq. (1.6) reduces to

$$\begin{aligned} \dot{y}(t) &= (A + V(t) - \lambda_k(t) \cdot Id) \cdot y(t) + L(t, \exp\{w_k(t+\cdot, t) \\ &\quad + s_k(t+\cdot, t)\} \cdot y_t) - e_k^T \cdot L(t, \exp\{w_k(t+\cdot, t)\} \cdot e_k) \cdot y(t) \\ &= (A + V(t) - \lambda_k(t) \cdot Id) \cdot y(t) + G(t, y_t), \end{aligned} \tag{3.4}$$

where Id represents the $n \times n$ unit matrix, and $G(t, y_t) = \mathrm{col}(g_1(t, y_t), ..., g_n(t, y_t))$ with

$$\begin{aligned} g_i(t, y_t) &= e_i^T \cdot L(t, \exp\{w_k(t+\cdot, t) + s_k(t+\cdot, t)\} \cdot y_t) \\ &\quad - e_k^T \cdot L(t, \exp\{w_k(t+\cdot, t)\} \cdot e_k) \cdot y_i(t), \qquad i \neq k, \end{aligned} \tag{3.5}$$

$$\begin{aligned} g_k(t, y_t) &= e_k^T \cdot L(t, \exp\{w_k(t+\cdot, t)\} \\ &\quad \cdot (\exp\{s_k(t+\cdot, t)\} \cdot y_t - y_k(t)\, e_k)). \end{aligned} \tag{3.6}$$

Since $\|L(t, \cdot)\|$ is in L^p, we see that there exists a nonnegative function ρ in L^p, such that

$$|L(t, \varphi)| \leqslant \rho(t) \cdot \|\varphi\| \qquad \text{for} \quad t \geqslant 0, \varphi \text{ in } C([-r, 0], E^n). \tag{3.7}$$

And so, from (3.5)–(3.7) we deduce

$$|g_i(t, \varphi)| \leqslant 2M_1 \cdot \rho(t) \|\varphi\|, \qquad t \geqslant 0, \varphi \in C([-r, 0], E^n), \tag{3.8}$$

where

$$\begin{aligned} M_1 = \sup_{t \geqslant 0} \, & (\exp(w_k(t+ \cdot, t)) + \exp(s_k(t+ \cdot, t)) \\ & + \exp(w_k(t+ \cdot, t) + s_k(t+ \cdot, t))). \end{aligned} \tag{3.9}$$

By (3.1) and the assumptions (H_1), (H_2), and (H_3), it follows that M_1 is finite. So, from (3.8), $\|g_i(t, \cdot)\|$, $1 \leqslant i \leqslant n$, is in L^p, and so $\|G(t, \cdot)\|$ is in L^p. Obviously, $G(t, \cdot)$ is linear with the second variable for each $t \geqslant 0$.

Now, the second step of the proof is to decompose the variable $y = \mathrm{col}(y_1, ..., y_n) = \mathrm{col}(Y_1, Y_2, Y_3)$, with $Y_1 = \mathrm{col}(y_1, ..., y_{k-1})$ in E^{k-1}, $Y_2 = y_k$ in E^1, and $Y_3 = \mathrm{col}(y_{k+1}, y_{K+2}, ..., y_n)$ in E^{n-k}. Then, Eq. (3.4) becomes

$$\begin{aligned} \dot{Y}_1(t) &= A_1(t)\, Y_1(t) + G_{11}(t, Y_{1_t}) + G_{12}(t, Y_{2_t}) + G_{13}(t, Y_{3_t}), \\ \dot{Y}_2(t) &= G_{21}(t, Y_{1_t}) + G_{22}(t, Y_{2_t}) + G_{23}(t, Y_{3_t}), \\ \dot{Y}_3(t) &= A_3(t)\, Y_3(t) + G_{31}(t, Y_{1_t}) + G_{32}(t, Y_{2_t}) + G_{33}(t, Y_{3_t}), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} A_1(t) &= \mathrm{diag}\{\lambda_1(t) - \lambda_k(t), ..., \lambda_{k-1}(t) - \lambda_k(t)\}, \\ A_3(t) &= \mathrm{diag}\{\lambda_{k+1}(t) - \lambda_k(t), ..., \lambda_n(t) - \lambda_k(t)\}, \end{aligned}$$

$G_{ij}(t, Y_{j_t}) = G_i(t, Q_j^T Y_{j_t})$ $(i, j = 1, 2, 3)$ with $Q_1 = (Id_{(k-1) \times (k-1)}, 0, 0)$, a $(k-1) \times n$ matrix, Q_1^T representing the transpose of Q_1, $Q_2 = e_k^T$, $Q_3 = (0, 0, Id_{(n-k) \times (n-k)})$, a $(n-k) \times n$ matrix, and $G_i(t, \cdot) = Q_i \cdot G(t, \cdot)$ $(i = 1, 2, 3)$. Obviously, $G_{ij}(t, \cdot)$ $(i, j = 1, 2, 3)$ is linear with respect to its second variable. Moreover, from (3.8) it readily follows

$$\|G_{ij}(t, \cdot)\| \leqslant M_2 \cdot \rho(t) \qquad (M_2 = 2nM_1), \; t \geqslant 0. \tag{3.11}$$

This implies $\|G_{ij}(t, \cdot)\|$ $(i, j = 1, 2, 3)$ is in L^p.

Consequently, by the above two steps we see that if the following theorem is proved, our theorem will be completed.

THEOREM 1′. *For each integer k, $1 \leqslant k \leqslant n$, there exists a number $t_k \geqslant 0$, such that for any $t_0 \geqslant t_k$ and each (Y_{1t_0}, Y_{2t_0}) in $C([-r, 0], E^{k-1}) \times C([-r, 0], E^1)$, there exists at least one solution $(Y_1(t_0, Y_{1t_0}, Y_{2t_0}), Y_2(t_0, Y_{1t_0}, Y_{2t_0}), Y_3(t_0, Y_{1t_0}, Y_{2t_0})) = (Y_1, Y_2, Y_3)$ of Eq. (3.10), which is defined on $[t_0 - r, +\infty)$ and satisfies $(Y_{1t_0}(t_0, Y_{1t_0}, Y_{2t_0}), Y_{2t_0}(t_0, Y_{1t_0}, Y_{2t_0})) = (Y_{1t_0}, Y_{2t_0})$ such that*

$$\lim_{t \to +\infty} (Y_1(t), Y_2(t), Y_3(t)) = (0, b, 0), \tag{3.12}$$

where b is a constant dependent on (Y_1, Y_2, Y_3). Moreover, there exists a number $\bar{t}_k \geqslant 0$, such that for any $t_0 \geqslant \bar{t}_k$ and any given constant b, there exists a solution $(Y_1(t_0, k, b), Y_2(t_0, k, b), Y_3(t_0, k, b)) = (\bar{Y}_1, \bar{Y}_2, \bar{Y}_3)$ of (3.10), which is defined on $[t_0 - r, +\infty)$ such that

$$\lim_{t \to +\infty} (\bar{Y}_1(t), \bar{Y}_2(t), \bar{Y}_3(t)) = (0, b, 0). \tag{3.13}$$

We shall employ the alternative method to prove Theorem 1′. Indeed, we consider system (3.10) as two systems:

$$\begin{aligned} \dot{Y}_1(t) &= A_1(t)\, Y_1(t) + G_{11}(t, Y_{1t}) \\ &\quad + G_{12}(t, Y_{2t}) + G_{13}(t, Y_{3t}), \\ \dot{Y}_2(t) &= G_{21}(t, Y_{1t}) + G_{22}(t, Y_{2t}) + G_{23}(t, Y_{3t}), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \dot{Y}_3(t) &= A_3(t)\, Y_3(t) + G_{21}(t, Y_{1t}) \\ &\quad + G_{32}(t, Y_{2t}) + G_{33}(t, Y_{3t}). \end{aligned} \tag{3.15}$$

Now, let $(Y_1(t_0, Y_{1t_0}, Y_{2t_0}, Y_3), Y_2(t_0, Y_{1t_0}, Y_{2t_0}, Y_3))$ denote the solution of (3.14) for a given function Y_3 in $C([t_0 - r, +\infty), E^{n-k})$ and initial data (Y_{1t_0}, Y_{2t_0}) in $C([-r, 0], E^{k-1}) \times C([-r, 0], E^1)$. Then, Eq. (3.15) becomes the equation of Y_3:

$$\begin{aligned} \dot{Y}_3(t) &= A_3(t)\, Y_3(t) + G_{31}(t, Y_{1t}(t_0, Y_{1t_0}, Y_{2t_0}, Y_3)) \\ &\quad + G_{32}(t, Y_{2t}(t_0, Y_{1t_0}, Y_{2t_0}, Y_3)) + G_{33}(t, Y_{3t}). \end{aligned} \tag{3.16}$$

It is clear that if Y_3 is a solution of (3.16), then $(Y_1(t_0, Y_{1t_0}, Y_{2t_0}, Y_3), Y_2(t_0, Y_{1t_0}, Y_{2t_0}, Y_3), Y_3)$ is a solution of (3.10), satisfying that $Y_{1t_0}(t_0, Y_{1t_0}, Y_{2t_0}, Y_3) = Y_{1t_0}$, $Y_{2t_0}(t_0, Y_{1t_0}, Y_{2t_0}, Y_3) = Y_{2t_0}$. We shall see by the following two propositions that in such solutions there exist those as stated in Theorem 1′. Now, we state these propositions:

PROPOSITION 1. *Assume that* Y_3 *is in* $C([t_0-r, +\infty), E^{n-k})$ *and satisfies*

$$W(Y_3)=\left(\int_{t_0+r}^{\infty} \|\bar{Y}_{3t}\|^q\, dt\right)^{1/q} < +\infty, \tag{3.17}$$

where $\|\bar{Y}_{3t}\| = \sup_{-2r\leqslant s\leqslant 0} |Y_3(t+s)|$.

Let (Y_1, Y_2) *be a solution of* (3.14) *with respect to this* Y_3. *Then,* $\lim_{t\to+\infty} Y_1(t)=0$, *and* $\lim_{t\to+\infty} Y_2(t)$ *exists. Moreover, there exists a number* $\bar{t}'_k \geqslant 0$, *which is independent of* Y_3, *such that for any* $t_0 \geqslant \bar{t}'_k$ *and any constant* b, *there exists a solution* $(Y_1(t_0, k, b, Y_3), Y_2(t_0, k, b, Y_3))$ *of* (3.14), *which is defined on* $[t_0-r, +\infty)$, *such that*

$$\lim_{t\to+\infty} (Y_1(t_0, k, b, Y_3)(t), Y_2(t_0, k, b, Y_3)(t)) = (0, b).$$

Before stating Proposition 2, we introduce the space $S(t_0)$ for $t_0 \geqslant 0$ as

$$S(t_0)=\{Y_3\colon Y_3 \text{ in } C([t_0-r, +\infty), E^{n-k}) \text{ and satisfying (3.17)}\}$$

with the norm $W(Y_3)$ defined by (3.17).

PROPOSITION 2. *There exists a number* $t'_k \geqslant 0$, *such that for any* $t_0 \geqslant t'_k$ *and each* (Y_{1t_0}, Y_{2t_0}) *in* $C([-r, 0], E^{k-1})\times C([-r, 0], E^1)$ *there exists at least one function* Y_3 *in* $S(t_0)$, *such that it satisfies Eq.* (3.16) *and* $\lim_{t\to+\infty} Y_3(t)=0$. *Such functions we denote by* $Y_3(t_0, Y_{1t_0}, Y_{2t_0})$. *Moreover, there exists a number* $\bar{t}''_k \geqslant \bar{t}'_k$, *such that for any* $t_0 \geqslant \bar{t}''_k$ *and any constant* b, *there exists at least one function* $Y_3(t_0, k, b)$ *in* $S(t_0)$, *such that it satisfies Eq.* (3.16) *as in which* $(Y_1, Y_2)=(Y_1(t_0, k, b, Y_3), Y_2(t_0, k, b, Y_3))$, *where* $(Y_1(t_0, k, b), Y_2(t_0, k, b))$ *is given in Proposition* 1, *and* $\lim_{t\to+\infty} Y_3(t_0, k, b)(t)=0$.

If these two propositions are proved, the proof of Theorem 1′ is immediately completed. For in it, let $t_k = t'_k$ and $\bar{t}_k = \bar{t}''_k$. We see that for any $t_0 \geqslant t_k$ (resp. $t_0 \geqslant \bar{t}_k$), the vector functions $(Y_1(t_0, Y_{1t_0}, Y_{2t_0}, Y_3(t_0, Y_{1t_0}, Y_{2t_0})), Y_2(t_0, Y_{1t_0}, Y_{2t_0}, Y_3(t_0, Y_{1t_0}, Y_{2t_0})), Y_3(t_0, Y_{1t_0}, Y_{2t_0}))$ (resp. $(Y_1(t_0, k, b, Y_3(t_0, k, b)), Y_2(t_0, k, b, Y_3(t_0, k, b)), Y_3(t_0, k, b)))$ given by Propositions 1 and 2 are just the solutions required in Theorem 1′. That is to say they are solutions of (3.10) and satisfy (3.12) (resp. (3.13)). Thus, the rest of the proof is to prove Propositions 1 and 2.

We first show Proposition 1. The proof is based on the application of Corollary 1 and Lemma 5. For this, we rewrite Eq. (3.14) for $t \geqslant t_0 + r$ as

$$\begin{aligned} \dot{Y}_1(t) &= M_1(t, \bar{Y}_{1t}) + N_1(t, \bar{Y}_{2t}) + R_1(t), \\ \dot{Y}_2(t) &= P_1(t, \bar{Y}_{1t}) + Q_1(t, \bar{Y}_{2t}) + R_2(t), \end{aligned} \tag{3.18}$$

where $\bar{Y}_1(t) = Y_2(t)$ and $\bar{Y}_2(t) = Y_1(t)$ for $t \geqslant t_0 - r$, $\bar{Y}_{it}$ $(i = 1, 2)$ denotes the translation over $[-2r, 0]$ for $t \geqslant t_0 + r$, $M_1(t, \cdot)$ and $P_1(t, \cdot)$ (resp. $N_1(t, \cdot)$ and $Q_1(t, \cdot)$) are functionals on the space $C([-2r, 0], E^1)$ (resp. $C([-2r, 0], E^{k-1})$) for $t \geqslant t_0 + r$, R_1 and R_2 are in $C([t_0 + r, +\infty), E^1)$ and $C([t_0 + r, +\infty), E^{k-1})$. Comparing this equation to the original Eq. (3.14), we have

LEMMA 7. *$M_1(t, \cdot)$ (resp. $N_1(t, \cdot)$) are bounded linear functionals on $C([-2r, 0], E^1)$ (resp. $C([-2r, 0], E^{k-1})$) for each $t \geqslant t_0 + r$, and R_1 is in $C([t_0 + r, +\infty), E^1)$. Moreover, $\|M_1(t, \cdot)\|$ and $|R_1(t)|$ are in L^1 and $\|N_1(t, \cdot)\|$ is in L^p.*

Let us prove this lemma. From the second equation of (3.14) we see that

$$\begin{aligned} G_{22}(t, Y_{2t}) &= G_2(t, Q_2^T \cdot Y_{2t}) \\ &= Q_2 \cdot G(t, Q_2^T \cdot Y_{2t}) = g_k(t, y_{kt} \cdot e_k) \\ &= e_k^T \cdot L(t, \exp\{w_k(t + \cdot, t)\} \\ &\quad \cdot (\exp\{s_k(t + \cdot, t)\} \cdot y_{kt} e_k - y_k(t)\, e_k)). \end{aligned}$$

Now let

$$h_k(t, s) = \exp(s_k(t + s, t)) - 1 \qquad \text{for} \quad -r \leqslant s \leqslant 0. \tag{3.19}$$

Then,

$$\begin{aligned} G_{22}(t, Y_{2t}) &= e_k^T \cdot L(t, \exp(w_k(t + \cdot, t)) \cdot h_k(t, \cdot) \cdot y_{kt} \cdot e_k) \\ &\quad + e_k^T \cdot L(t, \exp(w_k(t + \cdot, t)) \cdot (y_{kt} - y_k(t)) \cdot e_k) \\ &= e_k^T \cdot L(t, \exp(w_k(t + \cdot, t)) \cdot h_k(t, \cdot) \cdot Q_2^T \cdot Y_{2t}) \\ &\quad + e_k^T \cdot L(t, \exp(w_k(t + \cdot, t)) \cdot Q_2^T \cdot (Y_{2t} - Y_2(t))). \end{aligned}$$

Noticing that $Y_{2t} - Y_2(t) = \int_t^{t+\cdot} dY_2/ds \cdot ds$ for $t \geqslant t_0 + r$ and the second equation of (3.14), we have

$$\begin{aligned} G_{22}(t, Y_{2t}) &= e_k^T \cdot L(t, \exp(w_k(t + \cdot, t)) \cdot h_k(t, \cdot) \cdot Q_2^T \cdot Y_{2t}) \\ &\quad + e_k^T \cdot L\left(t, \exp(w_k(t + \cdot, t)) \cdot Q_2^T \cdot \int_t^{t+\cdot} G_{22}(s, Y_{2s})\, ds\right) \\ &\quad + e_k^T \cdot L\left(t, \exp(w_k(t + \cdot, t)) \cdot Q_2^T \cdot \int_t^{t+\cdot} G_{21}(s, Y_{1s})\, ds\right) \\ &\quad + e_k^T \cdot L\left(t, \exp(w_k(t + \cdot, t)) \cdot Q_2^T \int_t^{t+\cdot} G_{23}(s, Y_{3s})\, ds\right). \end{aligned}$$

So, comparing the first equation of (3.18) to the second equation of (3.14), we get

$$M_1(t, \bar{Y}_{1t}) = e_k^T \cdot L(t, \exp(w_k(t+\cdot, t)) \cdot h_k(t, \cdot) \cdot Q_2^T \cdot Y_{2t})$$

$$+ e_k^T \cdot L\left(t, \exp(w_k(t+\cdot, t)) \cdot Q_2^T \cdot \int_t^{t+\cdot} G_{22}(s, Y_{2s})\, ds\right),$$

$$N_1(t, \bar{Y}_{2t}) = e_k^T \cdot L\left(t, \exp(w_k(t+\cdot, t)) \cdot Q_t^{T+\cdot}\, G_{21}(s, Y_{1s})\, ds\right)$$

$$+ G_{21}(t, Y_{1t}),$$

$$R_1(t) = e_k^T \cdot L\left(t, \exp(w_k(t+\cdot, t)) \cdot Q_2^T \cdot \int_t^{t+\cdot} G_{23}(s, Y_{3s})\, ds\right)$$

$$+ G_{23}(t, Y_{3t}).$$

So the first part of Lemma 7 is clear. Moreover, by (3.11) we get

$$\begin{aligned} |M_1(t, \bar{Y}_{1t})| &\leqslant M_2 \cdot \rho(t) \cdot \left(\|h_k(t, \cdot)\| + M_2 \cdot \int_{t-r}^{t} \rho(s)\, ds\right) \cdot \|\bar{Y}_{1t}\|, \\ |N_1(t, \bar{Y}_{2t})| &\leqslant M_2 \cdot \rho(t) \cdot \left(1 + M_2 \cdot \int_{t-r}^{t} \rho(s)\, ds\right) \cdot \|\bar{Y}_{2t}\|, \\ |R_1(t)| &\leqslant M_2 \cdot \rho(t) \cdot \left(1 + M_2 \int_{t-r}^{t} \rho(s)\, ds\right) \cdot \|\bar{Y}_{3t}\|. \end{aligned} \tag{3.20}$$

Since $\|L(t, \cdot)\|$ is in L^p, it follows that δ_k is in L^p from (3.1), and so by (3.19) and Lemma 3, $\|h_k(t, \cdot)\|$ is in L^q. Noticing that $p \leqslant 2 \leqslant q$ and that $\rho \in L^p$ implies that $\int_{t-r}^{t} \rho(s)\, ds$ is in L^p and $\int_{t-r}^{t} \rho(s)\, ds \to 0$ as $t \to +\infty$, we can conclude that $\int_{t-r}^{t} \rho(s)\, ds$ is in L^q and bounded. By these facts and Hölder's inequality, we get immediately that $\|M_1(t, \cdot)\|$ is in L^1 and $\|N_1(t, \cdot)\|$ is in L^p. Finally, by (3.17) we also get R_1 is in L^1. Thus, the proof of this lemma is completed.

LEMMA 8. *$P_1(t, \cdot)$ (resp. $Q_1(t, \cdot)$) are bounded linear functionals on $C([-2r, 0], E^1)$ (resp. $C([-2r, 0], E^{k-1})$) for each $t \geqslant t_0 + r$, and R_2 is in $C([t_0 + r, +\infty), E^{k-1})$. Moreover, $\|P_1(t, \cdot)\|$ is in L^p, $|R_2(t)|$ is in L^1, and the equation*

$$\dot{\bar{Y}}_2(t) = Q_1(t, \bar{Y}_{2t}) \tag{3.21}$$

is exponentially stable.

The proof of this lemma is straightforward. From the first equation of (3.14) we get

$$P_1(t, \bar{Y}_{1t}) = G_{12}(t, Y_{2t}), \qquad R_2(t) = G_{13}(t, Y_{3t}),$$
$$Q_1(t, \bar{Y}_{2t}) = A_1(t)\, Y_1(t) + G_{11}(t, Y_{1t}),$$

and so by (3.11)

$$|P_1(t, \bar{Y}_{1t})| \leqslant M_2 \cdot \rho(t)\, \|\bar{Y}_{1t}\|, \qquad |R_2(t)| \leqslant M_2 \cdot \rho(t)\, \|\bar{Y}_{3t}\|,$$

and so, $\|P_1(t, \cdot)\|$ is in L^p and R_1 is in L^1 by (3.17). Since, by the proof of Lemma 2, the equation $\dot{\bar{Y}}_2(t) = A_1(t)\, \bar{Y}_2(t)$ is exponentially stable (with any $-\alpha$, $0 < \alpha < \min_{1 \leqslant i \leqslant k-1} \{\lambda_k - \lambda_i\}$, as its asymptotic exponent) and $\|G_{21}(t, \cdot)\|$ is in L^p, we can conclude that Eq. (3.21) is also exponentially stable and its asymptotic exponent can be any $-\alpha$ for $0 < \alpha < \min_{1 \leqslant i \leqslant k-1}$ $\{\lambda_k - \lambda_i\}$. This completes the second part of this lemma. The first part is obvious and so the lemma is proved.

The conclusion from these two lemmas is that Eq. (3.18) verifies the conditions of Lemma 4, and, more specifically, its corollary. Therefore the conclusions of Corollary 1 hold:

$$\lim_{t \to +\infty} \bar{Y}_1(t) \text{ exists} \qquad \text{and} \qquad \lim_{t \to +\infty} \bar{Y}_2(t) = 0,$$

which implies the first part of Proposition 1, and the surjectivity holds, for t_0 large enough, with respect to the data in $C([-2r, 0], E^1)$ for Eq. (3.18). Since the solutions of (3.14) constitute only a subset of this set, we must then show that there is still surjectivity with respect to the solutions of Eq. (3.14). The reason for this is to consider special data for Eq. (3.14). We take $Y_{1t_0} = 0$, $Y_{2t_0} = c$, where c is a constant. Now let

$$(\bar{Y}_1, \bar{Y}_2)/[t_0 - r, t_0] = (c, 0),$$
$$(\bar{Y}_1, \bar{Y}_2)/[t_0, t_0 + r] = (Y_2, Y_1)/[t_0, t_0 + r],$$

where $(Y_1, Y_2) = (Y_1(t_0, Y_{1t_0}, Y_{2t_0}, Y_3), Y_2(t_0, Y_{1t_0}, Y_{2t_0}, Y_3))$. This gives a set of data for Eq. (3.18).

On te other hand, from (3.18) and (3.14) we have

$$\lim_{t \to +\infty} \bar{Y}_1(t) = \bar{Y}_1(t_0 + r) + \int_{t_0 + r}^{\infty} M_1(s, \bar{Y}_{1s})\, ds$$
$$+ \int_{t_0 + r}^{\infty} N_1(s, \bar{Y}_{2s})\, ds + \int_{t_0 + r}^{\infty} R_1(s)\, ds,$$

and

$$\bar{Y}_1(t_0+r) = Y_2(t_0) + \int_{t_0}^{t_0+r} G_{21}(s, Y_{1s})\,ds + \int_{t_0}^{t_0+r} G_{22}(s, Y_{2s})\,ds + \int_{t_0}^{t_0+r} G_{23}(s, Y_{3s})\,ds$$

and so

$$\lim_{t\to+\infty} Y_2(t) = c + w_1(t_0, c, Y_3) + \int_{t_0}^{t_0+r} G_{23}(s, Y_{3s})\,ds + \int_{t_0+r}^{\infty} R_1(s)\,ds, \tag{3.22}$$

where

$$w_1(t_0, c, Y_3) = \int_{t_0}^{t_0+r} G_{21}(s, Y_{1s})\,ds + \int_{t_0}^{t_0+r} G_{22}(s, Y_{2s})\,ds + \int_{t_0+r}^{\infty} M_1(s, \bar{Y}_{1s})\,ds + \int_{t_0+r}^{\infty} N_1(s, \bar{Y}_{2s})\,ds. \tag{3.23}$$

Now we show that there exists a nonnegative function d_1 of t_0, which is independent of c and Y_3, such that $d_1(t_0) = o(1)$ as $t_0 \to +\infty$ and

$$|w_1(t_0, c, Y_3) - w_1(t_0, c', Y_3)| \leqslant d_1(t_0)\,|c-c'|. \tag{3.24}$$

Let $(Y_1, Y_2) = (Y_1(t_0, 0, c, Y_3), Y_2(t_0, 0, c, Y_3))$, $(Y_1', Y_2') = (Y_1(t_0, 0, c', Y_3), Y_2(t_0, 0, c', Y_3))$, $(\bar{Y}_1(t), \bar{Y}_2(t)) = (Y_2(t), Y_1(t))$, $(\bar{Y}_1'(t), \bar{Y}_2'(t)) = (Y_2'(t), Y_1'(t))$ for $t \geqslant t_0 - r$. Further let, by (3.20), for $t \geqslant t_0 + r$,

$$|M_1(t,\varphi)| \leqslant m_1(t)\cdot\|\varphi\|, \qquad |P_1(t,\varphi)| \leqslant p_1(t)\cdot\|\varphi\|, \qquad |N_1(t,\psi)| \leqslant n_1(t)\,\|\psi\|,$$

with m_1 in L^1 and p_1 and n_1 in L^p.

And so,

$$\text{the left side of (3.24)} \leqslant M_2\cdot\int_{t_0}^{t_0+r} \rho(s)\cdot(\|Y_{1s}-Y_{1s}'\| + \|Y_{2s}-Y_{2s}'\|)\,ds + \int_{t_0+r}^{\infty} m_1(s)\cdot\|\bar{Y}_{1s}-\bar{Y}_{1s}'\|\,ds + \int_{t_0+r}^{\infty} n_1(s)\,\|\bar{Y}_{2s}-\bar{Y}_{2s}'\|\,ds. \tag{3.25}$$

Noticing that $(Y_1(t)-Y_1'(t), Y_2(t)-Y_2'(t))$ is a solution of (3.14) for $t\geqslant t_0$ with $Y_3=0$ and $(\bar{Y}_1(t)-\bar{Y}_1'(t), \bar{Y}_2(t)-\bar{Y}_2'(t))$ is a solution of (3.18) for $t\geqslant t_0+r$ with $R_1=R_2=0$, so, by estimates (2.17) and (2.13), (2.14), we deduce

$$\|Y_{1s}-Y_{1s}'\|+\|Y_{2s}-Y_{2s}'\|\leqslant K_3\cdot|c-c'|, \qquad t_0\leqslant s\leqslant t_0+r,$$

the sum of the last two terms in the right side of (3.25)

$$\leqslant d(t_0+r)\cdot(\|\bar{Y}_{1t_0+r}-Y_{1t_0+r}'\|+\|\bar{Y}_{2t_0+r}-\bar{Y}_{2t_0+r}'\|),$$

where $K_3=\sup_{t_0\geqslant 0}\{K_1(t_0)+K_2(t_0)\}$, $K_1(t_0)$ and $K_2(t_0)$ are given by Lemma 5, and $d(t_0)$ is given by (2.14) (of course, m, n, and p in Lemma 4 and Lemma 5 are replaced by m_1, n_1, and p_1, respectively). And so

$$\text{the left side of (3.24)}\leqslant\left(M_2K_3\cdot\int_{t_0}^{t_0+r}\rho(s)\,ds+K_3d(t_0+r)\right)\cdot|c-c'|$$

and so, if we let

$$d_1(t_0)=K_3\left(M_2\int_{t_0}^{t_0+r}\rho(s)\,ds+d_3(t_0+r)\right), \tag{3.26}$$

we see that $d_1(t_0)\to 0$ as $t_0\to+\infty$ and satisfies (3.24).

Now let β be a given number, $0<\beta<1$. Then, there exists a constant $\bar{t}_k'\geqslant 0$, such that for $t_0\geqslant\bar{t}_k'$, $d_1(t_0)<\beta$. So, for $t_0\geqslant\bar{t}_k'$ and each Y_3 in $S(t_0)$, the mapping defined by the right side of (3.22) from E^1 to E^1 is an isomorphism. This implies the surjectivity with respect to the solutions of Eq. (3.14), which completes the proof of Proposition 1.

Remark 3.2. When applying (2.5), (2.6), and (2.17) to the solutions of Eq. (3.18) and Eq. (3.14), we can get that there exist a constant $K_1\geqslant 0$, which is independent of t_0 and Y_3 in $S(t_0)$, and a function d_2 defined on R^+, which is independent of Y_3 in $S(t_0)$ and $d_2(t_0)=o(1)$ as $t_0\to+\infty$, such that for any solution (Y_1, Y_2) of (3.14) with initial data (Y_{1t_0}, Y_{2t_0}), the following inequalities hold for $i=1, 2$:

$$\begin{aligned}\|Y_{it}\|&\leqslant K_1\cdot(\|Y_{1t_0}\|+\|Y_{2t_0}\|)+d_2(t_0)\cdot W(Y_3), &&t_0\leqslant t\leqslant t_0+r,\\ \|\bar{Y}_{it}\|&\leqslant K_1\cdot(\|Y_{1t_0}\|+\|Y_{2t_0}\|)+d_2(t_0)\cdot W(Y_3), &&t\geqslant t_0+r.\end{aligned} \tag{3.27}$$

Remark 3.3. Let

$$w_2(t_0, c, Y_3)=c+w_1(t_0, c, Y_3)+\int_{t_0}^{t_0+r}G_{23}(s, Y_{3s})\,ds+\int_{t_0+r}^{\infty}R_1(s)\,ds.$$

From the proof of Proposition 1 we see that for each $t_0\geqslant\bar{t}_k'$ and each Y_3

in $S(t_0)$, the mapping $w_2(t_0, \cdot, Y_3)$ from E^1 to E^1 is an isomorphism. Let $t_0 \geqslant \bar{t}'_k$ and Y_3, Y'_3 in $S(t_0)$ be given. Then for each b in E^1, there exist c and c' in E^1 such that

$$w_2(t_0, c, Y_3) = b = w_2(t_0, c', Y'_3).$$

Using (3.27) we can obtain that there exists a nonnegative function d_3 defined on $[\bar{t}'_k, +\infty)$, which is independent of Y_3 in $S(t_0)$, such that $d_3(t_0) = o(1)$ as $t \to +\infty$ and

$$|c - c'| \leqslant d_3(t_0) \cdot W(Y_3 - Y'_3), \tag{3.28}$$

where $t_0 \geqslant \bar{t}'_k$ and Y_3, Y'_3 in $S(t_0)$.

We turn now to the proof of Proposition 2.

Let (Y_{1t_0}, Y_{2t_0}) be in $C([-r, 0], E^{k-1}) \times C([-r, 0], E^1)$ and $t_0 \geqslant 0$. Now we define an operator $T = T(t_0, Y_{1t_0}, Y_{2t_0})$ on $S(t_0)$ as

$$\begin{aligned}(TY_3)(t) &= -\int_t^\infty \exp\left(\int_s^t \Lambda_3(\zeta)\, d\zeta\right) \cdot (G_{31}(s, Y_{1s}) + G_{32}(s, Y_{2s}) \\ &\qquad + G_{33}(s, Y_{3s}))\, ds, \qquad t \geqslant t_0, \\ (TY_3)(t) &= (TY_3)(t_0), \qquad t_0 - r \leqslant t \leqslant t_0,\end{aligned} \tag{3.29}$$

where $(Y_1, Y_2) = (Y_1(t_0, Y_{1t_0}, Y_{2t_0}, Y_3), Y_2(t_0, Y_{1t_0}, Y_{2t_0}, Y_3))$ is a solution of (3.14). We shall show that T maps $S(t_0)$ into itself and is a contraction. For this, we make the following estimates.

Let β, $0 < \beta < \min_{k+1 \leqslant i \leqslant n} \{\lambda_i - \lambda_k\}$, be a constant. Then, from the proof of Lemma 2 we see that there exist a number $t''_k \geqslant 0$, and $K_2 \geqslant 0$, such that for $t \geqslant t''_k$,

$$\exp\left(\int_s^t \Lambda_3(\zeta)\, d\zeta\right) \leqslant K_2 \cdot e^{-\beta(s-t)}, \qquad s \geqslant t. \tag{3.30}$$

Let $t_0 \geqslant t''_k$. So, by (3.11) and (3.27), we have if $t \geqslant t_0 + r$,

$$\begin{aligned}|(TY_3)(t)| &\leqslant K_2 M_2 \cdot \int_t^\infty e^{-\beta(s-t)} \cdot \rho(s)(\|\bar{Y}_{1s}\| + \|\bar{Y}_{2s}\| + \|\bar{Y}_{3s}\|)\, ds \\ &\leqslant 2K_2 M_2 \int_t^\infty e^{-\beta(s-t)} \cdot \rho(s)[K_1 \cdot (\|Y_{1t_0}\| + \|Y_{2t_0}\|) \\ &\qquad + d_2(t_0) \cdot W(Y_3)]\, ds \\ &\qquad + \left(K_2 M_2 \left(\int_t^\infty e^{-\beta p(s-t)} \cdot \rho^p(s)\, ds\right)^{1/p}\right) \cdot W(Y_3) \\ &\leqslant d_4(t) \cdot (\|Y_{1t_0}\| + \|Y_{2t_0}\|) + d_5(t) \cdot W(Y_3),\end{aligned} \tag{3.31}$$

where $d_4(t)=2K_1K_2M_2\cdot\int_t^\infty e^{-\beta(s-t)}\cdot\rho(s)\,ds$,

$$d_5(t)=2K_2M_2M_2'\cdot\int_t^\infty e^{-\beta(s-t)}\cdot\rho(s)\,ds$$

$$+K_2M_2\left(\int_t^\infty e^{-\beta p(s-t)}\cdot\rho^p(s)\,ds\right)^{1/p},$$

where M_2' is constant, such that $d_2(t_0)\leqslant M_2'$ for t_0. By Lemma 1 we can see that d_4 and d_5 are in L^q, and $d_4(t)\to 0$, $d_5(t)\to 0$ as $t\to+\infty$. Moreover, $d_4(t+s)\leqslant e^{\beta s}\cdot d_4(t)$, $d_5(t+s)\leqslant e^{\beta s}\cdot d_5(t)$ for $s\geqslant 0$. If $t_0\leqslant t\leqslant t_0+r$, then

$$|(TY_3)(t)|\leqslant K_2M_2\cdot\int_t^{t_0+r} e^{-\beta(s-t)}\cdot\rho(s)(\|Y_{1s}\|+\|Y_{2s}\|+\|Y_{3s}\|)\,ds$$

$$+K_2M_2\cdot\int_{t_0+r}^\infty e^{-\beta(s-t)}\cdot\rho(s)(\|\bar{Y}_{1s}\|+\|\bar{Y}_{2s}\|+\|\bar{Y}_{3s}\|)\,ds.$$

Noting $(\int_{t_0}^{t_0+r}\|Y_{3s}\|^q\,ds)^{1/q}\leqslant W(Y_3)$, where $\|Y_{3s}\|=\sup_{-r\leqslant\theta\leqslant 0}|Y_3(s+\theta)|$, by (3.27) we can deduce

$$|(TY_3)(t)|\leqslant d_6(t_0)(\|Y_{1t_0}\|+\|Y_{2t_0}\|)+d_7(t_0)\cdot W(Y_3), \tag{3.32}$$

where $d_6(t_0)=e^{\beta r}\cdot d_4(t_0)$, $d_7(t_0)=(M_2'/K_1)\cdot e^{\beta r}\cdot d_4(t_0)+K_2M_2[(\int_{t_0}^{t_0+r}\rho^p(s)\,ds)^{1/p}+(\int_{t_0+r}^\infty\rho^p(s)\,ds)^{1/p}]$. It is clear that $d_6(t_0)=o(1)$, $d_7(t_0)=o(1)$ as $t_0\to+\infty$.

And so, for $t\geqslant t_0+3r$, by (3.31),

$$\begin{aligned}\|(\overline{TY}_3)_t\|&=\sup_{-2r\leqslant s\leqslant 0}\{|(TY_3(t+s))|\}\\&\leqslant e^{2\beta r}\cdot d_4(t-2r)(\|Y_{1t_0}\|+\|Y_{2t_0}\|)\\&\quad+e^{2\beta r}\cdot d_5(t-2r)\cdot W(Y_3);\end{aligned}$$

for $t_0+r\leqslant t\leqslant t_0+3r$, by (3.31), (3.32),

$$\begin{aligned}\|(\overline{TY}_3)_t\|&=\sup\{|(TY_3)(t+s)|\colon -2r\leqslant s\leqslant 0\}\\&\leqslant\sup\{|(TY_3)(t+s)|\colon t+s\geqslant t_0+r,-2r\leqslant s\leqslant 0\}\\&\quad+\sup\{|(TY_3)(t+s)|\colon t+s\leqslant t_0+r,-2r\leqslant s\leqslant 0\}\\&\leqslant\sup\{|(TY_3)(t+s)|\colon t+s\geqslant t_0+r,-2r\leqslant s\leqslant 0\}\\&\quad+\sup\{|(TY_3)(t+s)|\colon t_0-r\leqslant t+s\leqslant t_0+r,-2r\leqslant s\leqslant 0\}\\&\leqslant(d_5(t_0+r)\cdot e^{2\beta r}+d_7(t_0))\cdot W(Y_3)\\&\quad+(e^{2\beta r}\cdot d_4(t_0+r)+d_6(t_0))\cdot(\|Y_{1t_0}\|+\|Y_{2t_0}\|)\\&=d_8(t_0)(\|Y_{1t_0}\|+\|Y_{2t_0}\|)+d_9(t_0)\cdot W(Y_3).\end{aligned}$$

Then, we can deduce

$$W(TY_3)=\left(\int_{t_0+r}^{\infty}\|(\overline{TY}_3)_t\|^q\,dt\right)^{1/q}\leqslant\left(\int_{t_0+r}^{t_0+3r}\|(\overline{TY}_3)_t\|^q\,dt\right)^{1/q}$$
$$+\left(\int_{t_0+3r}^{\infty}\|(\overline{TY}_3)_t\|^q\,dt\right)^{1/q}\leqslant d_{10}(t_0)\cdot(\|Y_{1t_0}\|+\|Y_{2t_0}\|)$$
$$+d_{11}(t_0)\cdot W(Y_3), \tag{3.33}$$

where

$$d_{10}(t_0)=(2r)^{1/q}\cdot d_8(t_0)+e^{2\beta r}\cdot\left(\int_{t_0+3r}^{\infty}(d_4(t-2r))^q\cdot dt\right)^{1/q},$$
$$d_{11}(t_0)=(2r)^{1/q}\cdot d_9(t_0)+e^{2\beta r}\left(\int_{t_0+3r}^{\infty}(d_5(t-2r))^q\,dt\right)^{1/q}.$$

Since $d_8(t_0)\to 0$, $d_9(t_0)\to 0$ as $t_0\to +\infty$ and d_4, d_5 are in L^q, it follows that $d_{10}(t_0)\to 0$, $d_{11}(t_0)\to 0$ as $t_0\to +\infty$. So, TY_3 is in $S(t_0)$ for $t_0\geqslant t_k''$, and so T maps $S(t_0)$ into itself.

Now let $t_k'\geqslant t_k''$, such that for $t_0\geqslant t_k'$, $d_{11}(t_0)\leqslant 1/2$. Then, for each $t_0\geqslant t_k'$ and Y_3, Y_3' in $S(t_0)$, from (3.33), we have

$$W(TY_3-TY_3')\leqslant d_{11}(t_0)\cdot W(Y_3-Y_3')\leqslant(1/2)\cdot W(Y_3-Y_3').$$

This shows that T is a strict contraction on $S(t_0)$.

Consequently, by the Banach fixed point theorem, T has a unique fixed point $Y_3=Y_3(t_0, Y_{1t_0}, Y_{2t_0})$ in $S(t_0)$. This Y_3 provides a solution of Eq. (3.16). Moreover, by (3.31), it follows that $Y_3(t)\to 0$ as $t\to +\infty$. Thus, the proof of the first part of Proposition 2 is complete.

To show the second part, let $t_0\geqslant\max\{\bar{t}_k', t_k''\}$, where $\bar{t}_k'$ is given in Proposition 1, and b a given constant. Now let us define the operator $T=T(t_0, b)$ on $S(t_0)$ the same as given in (3.29). But in (3.29), $(Y_1, Y_2)=(Y_1(t_0, k, b, Y_3), Y_2(t_0, k, b, Y_3))$, which is given in Proposition 1 with initial data $(Y_{1t_0}, Y_{2t_0})=(c, 0)$, where c satisfies that $w_2(t_0, c, Y_3)=b$. By (3.33), T maps $S(t_0)$ into itself. And for Y_3 and Y_3' in $S(t_0)$, by (3.28) and (3.33), we deduce that

$$W(TY_3-TY_3')\leqslant d_{10}(t_0)\cdot|c-c'|+d_{11}(t_0)\cdot W(Y_3-Y_3')$$
$$\leqslant d_{12}(t_0)\cdot W(Y_3-Y_3'), \tag{3.34}$$

where $d_{12}(t_0)=d_{10}(t_0)\,d_3(t_0)+d_{11}(t_0)$. Obviously, $d_{12}(t_0)\to 0$ as $t_0\to +\infty$. So, there exists a number $\bar{t}_k''\geqslant\max\{\bar{t}_k', t_k''\}$, such that $d_{12}(t_0)\leqslant 1/2$ for $t_0\geqslant\bar{t}_k''$, and so for such t_0, T is a strict contraction on $S(t_0)$ and, therefore, has a unique fixed point $Y_3 = Y_3(t_0, k, b)$ in $S(t_0)$. This Y_3 provides a solution of Eq. (3.16) with $(Y_1, Y_2) = (Y_1(t_0, k, b, Y_3), Y_2(t_0, k, b, Y_3))$. By (3.31) we see that $Y_3(t) \to 0$ as $t \to +\infty$, and, thus, we complete the proof of the second part of Proposition 2.

Combining the above obtained results, we complete the proof of Theorem 1.

Remark 3.4. Using (2.5)–(2.8), (2.17), (3.22), (3.31)–(3.33), we can get

$$|c| \leqslant (1 - d_{13}(t_0))^{-1}(b + d_{14}(t_0)),$$

where $c = Y_2(t_0, k, b, Y_3(t_0, k, b))$, $d_{13}(t_0)$ and $d_{14}(t_0)$ are two functions defined for large t_0, independent of b, and that $d_{13}(t_0) = o(1)$, $d_{14}(t_0) = o(1)$ as $t_0 \to +\infty$.

Remark 3.5. We will not need Theorem 1 in its full generality but this result is of independent interest.

Remark 3.6. The second part of Theorem 1 can be proved directly by using a fixed point theorem to a proper operator.

In order to obtain the asymptotic formula for an arbitrary solution of (1.6), by Theorem 1, let us define an $n \times n$ matrix function X on $t \geqslant t_0 - r$, $t_0 \geqslant \bar{t}_0 = \max_{1 \leqslant k \leqslant n} \{\bar{t}_k\}$, as

$$X(t, t_0) = (x(t_0, 1, 1), ..., x(t_0, n, 1)).$$

Clearly, it is well defined and has the property that $x(t) = X(t, t_0) \cdot c$, $c \in E^n$, is a solution of Eq. (1.6). By the asymptotic formula (3.2) of $x(t_0, k, b)$ with $b = 1$, $1 \leqslant k \leqslant n$, as stated in Theorem 1, we have following

THEOREM 2. *There exists an $n \times n$ matrix function F defined on $t \geqslant t_0 - r$, $t_0 \geqslant \bar{t}_0$, $F(t, t_0) \to 0$ as $t \to +\infty$, and*

$$\sup_{t \geqslant t_0 - r} |F(t, t_0)| \to 0, \qquad \textit{as} \quad t_0 \to +\infty, \textit{ such that}$$

$$X(t, t_0) = (Id + F(t, t_0)) \cdot \exp\{W(t, t_0) + S(t, t_0)\}, \tag{3.35}$$

where W, S are defined in (3.1).

Proof. From the proof of Theorem 1, let $f_k(t, t_0) = \mathrm{col}(Y_1(t_0, k, 1, Y_3(t_0, k, 1))(t), Y_2(t_0, k, 1, Y_3(t_0, k, 1))(t) - 1, Y_3(t_0, k, 1)(t))$. It follows that f_k defines on $t \geqslant t_0 - r$, $t_0 \geqslant \bar{t}_0$ and varifies for $t \geqslant t_0 - r$, $t_0 \geqslant \bar{t}_0$,

$$x(t_0, k, 1)(t) = \exp\{w_k(t, t_0) + s_k(t, t_0)\} \cdot (e_k + f_k(t, t_0)).$$

Noticing $Y_{1t_0} = 0$ and in this case that (2.17) still holds with $K_2(t_0) = K \cdot K_1(t_0) \cdot \int_{t_0}^{t_0 + r} p(s)\, ds$, we see that $K_2(t_0) = o(1)$ as $t_0 \to +\infty$. So, by

(2.5)–(2.8), (2.17), (3.31), (3.32) and Remark 3.4, we can show that $\sup_{t \geqslant t_0 - r} |f_k(t, t_0)| \to 0$ as $t_0 \to +\infty$. Clearly, $f_k(t, t_0) \to 0$ as $t \to +\infty$ by Theorem 1. Now let

$$F = (f_1, ..., f_k, ..., f_n).$$

It is easy to check that this F satisfies the requirements of Theorem 2.

Remark 3.6. By Theorem 2, we see that $X(t, t_0)$ is nonsingular for t_0 large enough and $t \geqslant t_0 - r$. Without loss of generality, in what follows we assume that $X(t, t_0)$ is nonsingular for $t_0 \geqslant \bar{t}_0$, $t \geqslant t_0 - r$.

4. Asymptotic Formula for Arbitrary Solution

The reason for interest in the special solutions will now become clear. We are going to show that a n-parameter family of special solutions of (1.6) characterizes the asymptotic behavior of all solutions of (1.6) as $t \to +\infty$.

Notation. Let $x(t_0, \varphi)$ denote the solution of (1.6) defined on $t \geqslant t_0 - r$, satisfying $x_{t_0}(t_0, \varphi) = \varphi$.

THEOREM 3. *For each solution $x(t_0, \varphi)$ of (1.6) with $t_0 \geqslant \bar{t}_0 + r$, φ in $C([-r, 0], E^n)$, there exist a constant $c(t_0, \varphi)$ in E^n and a vector function $f(t_0, \varphi)$ defined on $t \geqslant t_0 - r$, $f(t_0, \varphi)(t) = o(e^{-\alpha t})$ as $t \to +\infty$ for any $\alpha > 0$, such that*

$$x(t_0, \varphi)(t) = X(t, t_0) \cdot (c(t_0, \varphi) + f(t_0, \varphi)(t)), \qquad t \geqslant t_0 - r,$$

where X is given by (3.35).

Proof. Let $x(t) = x(t_0, \varphi)(t)$, $t \geqslant t_0 - r$, $t_0 \geqslant \bar{t}_0$, $X(t) = X(t, t_0)$. Define, for $t \geqslant t_0 - r$,

$$z(t) = X^{-1}(t) \cdot x(t). \tag{4.1}$$

Then,

$$\dot{x}(t) = (d/dt)(X(t)) \cdot z(t) + X(t) \cdot \dot{z}(t).$$

By the definition of X and (1.6), we deduce

$$X(t) \cdot \dot{z}(t) = L(t, X_t \cdot (z_t - z(t))), \qquad t \geqslant t_0. \tag{4.2}$$

Let

$$m(t) = X(t) \cdot \dot{z}(t) \qquad \text{for} \quad t \geqslant t_0. \tag{4.3}$$

Then, by (3.7), (4.2) we have for $t \geqslant t_0 + r$

$$\begin{aligned}|m(t)| &= |L(t, X_t(z_t - z(t)))| \\ &\leqslant \rho(t) \cdot \sup_{-r \leqslant \theta \leqslant 0} |X(t+\theta) \cdot (z(t+\theta) - z(t))| \\ &= \rho(t) \cdot \sup_{-r \leqslant \theta \leqslant 0} \left| X(t+\theta) \cdot \int_t^{t+\theta} \dot{z}(s)\, ds \right| \\ &= \rho(t) \cdot \sup_{-r \leqslant \theta \leqslant 0} \left| \int_t^{t+\theta} X(t+\theta) \cdot X^{-1}(s) \cdot m(s)\, ds \right|,\end{aligned}$$

and so by (3.35) we obtain

$$\begin{aligned}|m(t)| \leqslant M_1^2 \cdot \rho(t) \sup_{-r \leqslant \theta \leqslant 0} \int_{t+\theta}^{t} &|\exp\{W(t+\theta, s) \\ &+ S(t+\theta, s)\}| \cdot |m(s)|\, ds,\end{aligned} \tag{4.4}$$

where

$$M_1 = \sup_{t \geqslant t_0 - r,\, t_0 \geqslant i_0} \{|Id + F(t, t_0)| + |(Id + F(t, t_0))^{-1}|\}.$$

Since $\sup_{s \geqslant t} (1 + s - t)^{-1} \cdot \int_t^s v_i(\zeta)\, d\zeta \to 0$ as $t \to +\infty$ and d_i is in L^p by (3.1), $1 \leqslant i \leqslant n$, we can easily conclude that there exist two constants λ, $\lambda > \max_{1 \leqslant i \leqslant n} |\lambda_i|$, and $M_2 \geqslant 0$, such that for $t \geqslant s$

$$|\exp(-W(t, s) - S(t, s))| \leqslant M_2 \cdot e^{\lambda \cdot (t-s)}. \tag{4.5}$$

So, we find by (4.4)

$$\begin{aligned}|m(t)| &\leqslant M_1^2 M_2 \cdot \rho(t) \sup_{-r \leqslant \theta \leqslant 0} \int_{t+\theta}^{t} e^{\lambda \cdot (s-t-\theta)} \cdot |m(s)|\, ds \\ &\leqslant M_1^2 M_2 \cdot \rho(t) \cdot \int_{t-r}^{t} e^{\lambda \cdot (s-t+r)} \cdot |m(s)|\, ds,\end{aligned}$$

and so, we deduce for $t \geqslant t_0 + r$

$$e^{\lambda(t-t_0)} \cdot |m(t)| \leqslant M_1^2 M_2 M_3 \cdot \rho(t) \cdot \int_{t-r}^{t} e^{\lambda(s-t_0)} \cdot |m(s)|\, ds,$$

where $M_3 = e^{\lambda r}$. Now let

$$R(t) = e^{\lambda \cdot (t-t_0)} \cdot |m(t)|, \qquad t \geqslant t_0.$$

It follows that

$$R(t) \leqslant M_1^2 M_2 M_3 \cdot \rho(t) \cdot \int_{t-r}^{t} R(s)\, ds, \qquad t \geqslant t_0 + r. \tag{4.6}$$

So, by (1) of Lemma 6 we get that for any $\alpha > 0$, $\int_{t_0}^{\infty} e^{\alpha t} \cdot R(t)\, dt < \infty$, and this implies that R is in L^1. Noting that

$$\begin{aligned} |\dot{z}(t)| = |X^{-1}(t) \cdot m(t)| \leqslant {} & |\exp(-W(t, t_0) \\ & - S(t, t_0)) \cdot (Id + F(t, t_0))^{-1}| \\ & \cdot |m(t)| \leqslant M_1 M_2 \cdot e^{\lambda(t - t_0)} \cdot |m(t)| = M_1 M_2 \cdot R(t), \end{aligned}$$

we get that $\int_{t_0}^{\infty} |\dot{z}(t)|\, dt < +\infty$. This assures the existence of $\lim_{t \to +\infty} z(t)$. Let

$$c(t_0, \varphi) = \lim_{t \to +\infty} z(t) \qquad \text{and} \qquad f(t_0, \varphi)(t) = z(t) - c(t_0, \varphi)$$

for $t \geqslant t_0 - r$. So,

$$|f(t_0, \varphi)(t)| \leqslant \int_t^{\infty} |\dot{z}(s)|\, ds \leqslant M_1 M_2 \cdot \int_t^{\infty} R(s)\, ds,$$

and so, by (2) of Lemma 6 it follows that for any $\alpha > 0, f(t_0, \varphi)(t) = o(e^{-\alpha t})$ as $t \to +\infty$. Clearly, for $t \geqslant t_0 - r$,

$$x(t_0, \varphi)(t) = X(t, t_0) \cdot z(t) = X(t, t_0) \cdot (c(t_0, \varphi) + f(t_0, \varphi)(t)).$$

This completes the proof of Theorem 3.

Theorem 3 gives the asymptotic formula of solutions $x(t_0, \varphi)$ of (1.6) for $t_0 \geqslant \bar{t}_0 + r$. The following theorem will give the asymptotic formula for all solutions $x(t_0, \varphi)$ of (1.6) for $t_0 \geqslant 0$.

THEOREM 4 (Main Result). *There exist an $n \times n$ matrix function F defined on $t \geqslant t_0 - r, t_0 \geqslant 0, F(t, t_0) \to 0$ as $t \to +\infty$ for $t_0 \geqslant 0$, and $\sup_{t \geqslant t_0 - r} |F(t, t_0)| \to 0$ as $t_0 \to +\infty$, such that for every solution $x(t_0, \varphi)$ of (1.6) with $t_0 \geqslant 0$, φ in $C([-r, 0], E^n)$, there exist a constant $c(t_0, \varphi)$ in E^n and a vector function $f(t_0, \varphi)$ defined on $t \geqslant t_0 - r, f(t_0, \varphi)(t) = o(e^{-\alpha t})$ as $t \to +\infty$ for any $\alpha > 0$, such that*

$$\begin{aligned} x(t_0, \varphi)(t) = {} & (Id + F(t, t_0)) \cdot \exp\{W(t, t_0) + S(t, t_0)\} \\ & \cdot (c(t_0, \varphi) + f(t_0, \varphi)(t)), \qquad t \geqslant t_0 - r. \end{aligned} \tag{4.7}$$

Moreover, for each c in E^n,

$$x(t) = (Id + F(t, t_0)) \cdot \exp\{W(t, t_0) + S(t, t_0)\} \cdot c,$$

$t \geqslant t_0 - r$, $t_0 \geqslant \bar{t}_0$, *gives a solution of Eq.* (1.6).

Proof. Let us define

$$F_1(t, t_0) = \begin{cases} F(t, t_0), & t \geqslant t_0 - r,\ t_0 \geqslant \bar{t}_0 + r, \\ F(t, \bar{t}_0 + r), & t \geqslant \bar{t}_0 + r,\ t_0 < \bar{t}_0 + r, \\ F(\bar{t}_0 + r, \bar{t}_0 + r), & t_0 - r \leqslant t \leqslant \bar{t}_0 + r,\ t_0 < \bar{t}_0 + r. \end{cases}$$

$$X_1(t, t_0) = (Id + F_1(t, t_0)) \cdot \exp\{W(t, t_0) + S(t, t_0)\},$$

$$c_1(t_0, \varphi) = \begin{cases} c(t_0, \varphi), & t_0 \geqslant \bar{t}_0 + r, \\ \exp\{W(t_0, \bar{t}_0 + r) + S(t_0, \bar{t}_0 + r)\} \cdot c(\bar{t}_0 + r, x_{\bar{t}_0 + r}(t_0, \varphi)), \\ \qquad t_0 < \bar{t}_0 + r, \end{cases}$$

$$f_1(t_0, \varphi)(t) = \begin{cases} f(t_0, \varphi)(t), & t \geqslant t_0 - r,\ t_0 \geqslant \bar{t}_0 + r, \\ \exp\{W(t_0, \bar{t}_0 + r) + S(t_0, \bar{t}_0 + r)\} \cdot f(\bar{t}_0 + r, x_{\bar{t}_0 + r}(t_0, \varphi)), \\ \qquad t \geqslant \bar{t}_0 + r,\ t_0 < \bar{t}_0 + r, \\ X_1^{-1}(t, t_0) \cdot x(t_0, \varphi)(t) - c_1(t_0, \varphi), \\ \qquad t_0 - r \leqslant t \leqslant \bar{t}_0 + r,\ t_0 < \bar{t}_0 + r, \end{cases}$$

where F, c, f are given by Theorem 2 and Theorem 3. By Theorem 2 and Theorem 3, it is easy to see that F_1, c_1, f_1 have the properties as stated in Theorem 4 for F, c, f. Let us still use F, c, f to denote F_1, c_1, f_1. Then we show the first part of Theorem 4.

By Theorem 1, the second part is clear. Thus, we complete the proof of Theorem 4.

Remark 4.1. When applying Theorem 4 to Eq. (1.2), it is easy to see that the obtained result is slightly stronger than the conjecture we mentioned in Section 1, for in (1.3) we get that not only $f(t) \to 0$ as $t \to +\infty$ but also $f(t) = o(e^{-\alpha t})$ as $t \to +\infty$ for any $\alpha > 0$.

Remark 4.2. In order to compare the asymptotic formula (4.7) to the weaker form (1.5), we express it in another form:

$$x(t_0, \varphi)(t) = (Id + F(t, t_0)) \cdot \exp(W(t, t_0) + S(t, t_0)) \cdot c(t_0, \varphi)$$
$$+ H(t_0, \varphi)(t) \cdot \mathrm{col}(\exp(W(t, t_0) + S(t, t_0))),$$

where $H(t_0, \varphi)$ is a matrix function defined by $H(t_0, \varphi)(t) = (Id + F(t, t_0)) \cdot \mathrm{diag}\{f_1(t_0, \varphi)(t), ..., f_n(t_0, \varphi)(t)\}$, here $f_i(t_0, \varphi)$, $1 \leqslant i \leqslant n$, denotes the ith component function, so $H(t_0, \varphi)(t) = o(e^{-\alpha t})$ as $t \to +\infty$ for any $\alpha > 0$;

$\operatorname{col}(\exp(W(t, t_0)+S(t, t_0)))$ denotes the vector with components equal to $\exp(w_i(t, t_0)+s_i(t, t_0))$.

In the general case, we can not easily compute $c(t_0, \varphi)$. However, we have the following:

THEOREM 5. *Each component $c_i(t_0, \cdot)$ of $c(t_0, \cdot)$, $t_0 \geqslant \bar{t}_0 + r$, is a nontrivial linear functional on $C([-r, 0], E^n)$. Thus, each $c_i(t_0, \varphi) \neq 0$, except for those φ in a subspace of $C([-r, 0], E^n)$ of codimension one. Moreover, $c(t_0, \cdot)$ is continuous.*

Proof. From the proof of Theorem 4, we need only to show the result for $t_0 \geqslant \bar{t}_0 + r$. Let t_0 be such a number. For each $t \geqslant t_0$, $x(t_0, \varphi)(t)$ considered as a functional of φ is linear on $C([-r, 0], E^n)$, so is $z(t)$, $z(t)$ is given by (4.1). Now let $t \to +\infty$ to conclude from the proof of Theorem 3 that each $c_i(t_0, \cdot)$ is a linear functional on $C([-r, 0], E^n)$. From Theorem 1 it follows that $c_i(t_0, \cdot)$ is not identically zero.

The bound for $c(t_0, \cdot)$ proceeds as follows. With the aid of the estimate for each solution $x(t_0, \varphi)$ of (1.6) (it is not difficult to prove by Bellman's inequality),

$$|x(t_0, \varphi)(t)| \leqslant N \cdot e^{K(t-t_0)} \|\varphi\|, \qquad t \geqslant t_0,$$

where N, K are constants independent of t_0, from (4.3) we get

$$\begin{aligned} |m(t)| &= |L(t, X_t(z_t - z(t))| \\ &\leqslant \rho(t) \sup_{-r \leqslant s \leqslant 0} |x(t+s) - X(t+s)\, X^{-1}(t)\, x(t)| \\ &\leqslant N \cdot \rho(t) \sup_{-r \leqslant s \leqslant 0} (1 + |X(s+t)\, X^{-1}(t)|) \cdot e^{K(t-t_0)} \|\varphi\|. \end{aligned} \tag{4.8}$$

Now, without loss of generality, assume t_0 satisfies that for $t \geqslant \bar{t}_0$, $e^{\alpha r} M_1^2 M_2 M_3 \cdot \int_t^{t+r} \rho(s)\, ds \leqslant 1/2$, where $\alpha > 0$. So, by (4.6) and Lemma 6 we get

$$\int_{t_0+r}^{\infty} e^{\alpha t} R(t)\, dt \leqslant 2M_1^2 M_2 M_3 \cdot \int_{t_0}^{t_0+r} R(t) \cdot \int_s^{s+r} e^{\alpha s} \rho(s)\, ds\, dt. \tag{4.9}$$

And so, by (4.8), (4.9), the definitions of X, m, and R, and ρ in L^p, we can deduce that there exists a constant $M_0 \geqslant 0$, such that

$$\int_t^{\infty} R(s)\, ds \leqslant M_0 e^{-\alpha t} \cdot \|\varphi\|, \qquad t \geqslant t_0 + r.$$

From this, we get

$$|z(t)-z(t_0+r| \leqslant \int_{t_0+r}^{\infty} |\dot{z}(s)|\, ds \leqslant M_1 M_2 \cdot \int_{t_0+r}^{\infty} R(s)\, ds$$
$$\leqslant M_0 M_1 M_2 \cdot e^{-\alpha(t_0+r)} \cdot \|\varphi\|.$$

Let $t \to +\infty$. Then

$$|c(t_0, \varphi)-z(t_0+r)| \leqslant M_0 M_1 M_2 \cdot e^{-\alpha(t_0+r)} \cdot \|\varphi\|, \tag{4.10}$$

and from this, we get

$$|c(t_0, \varphi)| \leqslant (M_0 M_1 M_2 \cdot e^{-\alpha(t_0+r)} + M_0') \cdot \|\varphi\|, \tag{4.11}$$

where M_0' is a positive constant, such that $|z(t_0+r)| \leqslant M_0' \cdot \|\varphi\|$. Then, we complete the proof of Theorem 5.

Remark 4.3. We point out that M_0 and M_0' can be chosen independent of t_0, so by (4.11) we get that $c(t_0, \cdot)$ is uniformly bounded with respect to t_0. Moreover, since

$$|f(t_0, \varphi)(t)| = |c(t_0, \varphi)-z(t)| \leqslant M_1 M_2 \cdot \int_t^{\infty} R(s)\, ds$$
$$\leqslant M_0 M_1 M_2 \cdot e^{-\alpha t} \cdot \|\varphi\|,$$

we conclude that

$$\sup_{t \geqslant t_0+r} |f(t_0, \cdot)(t)| = O(e^{-\alpha t_0}) \qquad \text{as} \quad t_0 \to +\infty$$

for any $\alpha > 0$. Obviously, $f(t_0, \cdot)(t)$ is a linear operator on $C([-r, 0], E^n)$ for each $t \geqslant t_0$.

In conclusion, we emphasize a striking difference between the results obtained here and the results in [4]. In this paper, we obtain a class of special solutions given in Theorem 1 (just as those obtained in [6, 8, 9] for Eq. (1.1)), but in [4], such solutions were not obtained (this fact was pointed out in the last part of [4]). This, on the other hand, indicates that the method used here is superior to the "inductive procedure" used in [4] (still, the method used in [4] for quasi-triangular systems is superior). While comparing the method used in [6, 8, 9], we think that our method is very natural. The reason we must have the second step (i.e., the proof of Theorem 3) to get the asymptotic formula for all solutions of Eq. (1.6) is that, for $r > 0$, the solutions of Eq. (1.6) form a infinite dimensional space.

Finally, it should be noted that in (H_3) we assume $1 \leqslant p \leqslant 2$. Now, an open problem is if we can relax it to $p > 0$ to obtain a similar result to Theorem 4 (this problem is valid for Eq. (1.1), see [8]). If one uses the same method as in [8] to prove it, one will find that the proof is very complicated. Therefore, it is necessary to find a new way to prove this problem.

ACKNOWLEDGMENTS

The author thanks Professor You Bingli for his very valuable advice and guidance during the course of this work. Also, the author thanks the referees for their many valuable suggestions, especially concerning Section 3 of this paper.

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