JOURNAL OF DIFFERENTIAL EQUATIONS **153**, 61–81 (1999)
Article ID jdeq.1998.3544, available online at http://www.idealibrary.com on IDEAL®

Solitons of the Two-Dimensional 3-Component Gauged Sigma Model

Shangbing Ai and Xinfu Chen

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
E-mail: sxast10+@pitt.edu; xinfu+@pitt.edu

Received October 1, 1997

In this paper we study a boundary value problem of a system of two second order ordinary differential equations arising from the field equations of the 3-component U(1) gauged sigma model. We get a sufficient condition for the existence and a sufficient condition for the nonexistence of solutions of the boundary value problem. © 1999 Academic Press

Key Words: gauged sigma model; boundary value problem; shooting method.

1. INTRODUCTION

In this paper we study the following boundary value problem

$$u'' + \frac{1}{r}\, u' - \left(\frac{m^2(1-v)^2}{r^2} + p\right) \sin u \cos u = 0 \qquad \text{for} \quad r > 0, \tag{1.1}$$

$$v'' - \frac{1}{r}\, v' + 2(1-v) \sin^2 u = 0 \qquad \text{for} \quad r > 0, \tag{1.2}$$

$$u(0) = 0, \qquad v(0) = 0, \tag{1.3}$$

$$u(\infty) = \pi, \qquad v'(\infty) = 0. \tag{1.4}$$

Here m is a positive integer and p is a positive constant. Problem (1.1)–(1.4) was derived in [1, 2] in the study of field equations of the 3-component U(1) gauged model ("the A3M model") with spontaneously broken Z(2) symmetry. Note here we use the notation (u, v) instead of (θ, α) used in [1], where $u = \pi - \theta$ and $v = \alpha$. The solutions of the above BVP, referred to as topological solitons with the "topological charge" m of the 2-dimensional A3M model, describe stable bound states of the unit 3-component "easy-axis" Heisenberg field and the Maxwell field. We refer the readers to [1, 2] and the references therein for more details and the physical background to the problem.

0022-0396/99 $30.00

Numerical investigation by Bogolubsky and Bogolubskaya [1] shows that, for small positive p, (1.1)–(1.4) has a unique solution with the property

$$u' > 0, \qquad v' > 0, \qquad v < 1 \qquad \text{for all} \quad r > 0. \tag{1.5}$$

The purpose of this paper is to give a rigorous discussion of the existence of such solutions. In fact, we prove the following

THEOREM 1.1. (1) *For every $m > 0$, if $p \geqslant 4m$, there is no solution to Problem* (1.1)–(1.5).

(2) *For every $m > 0$, there exists $p^* = p^*(m) > 0$ such that for every $p \in (0, p^*)$, there exists at least a solution to Problem* (1.1)–(1.5). *In addition, when $m > 1$,*

$$p^*(m) \geqslant 4m \, \frac{\displaystyle\int_0^{\pi/2} \cos^2 \frac{u}{2} \sin u \left(\tan \frac{u}{2} \right)^{2/m} du}{\displaystyle\int_0^{\pi} \sin u \left(\tan \frac{u}{2} \right)^{2/m} du}. \tag{1.6}$$

Remark 1.2. 1. We believe that for every positive m, (1.1)–(1.5) has no solution when $p \in (p^*(m), \infty)$.

2. We do not have uniqueness result, though numerical evidence in [1] shows this.

The existence part of the theorem is proved by a 2-dimensional topological shooting argument, where the shooting parameters c and d are given in the following lemma.

LEMMA 1.3. *Let $m > 0$ and $p \in \mathbb{R}$ be given. If (u, v) is a solution of* (1.1)–(1.3) *near $r = 0+$, then there exists a pair (c, d) such that (u, v) satisfies*

$$u(r) = cr^m + O(r^{m+2}), \qquad v(r) = dr^2 + O(r^{2m+2}) \qquad \text{as} \quad r \searrow 0. \tag{1.7}$$

On the other hand, for every $(c, d, m, p) \in \mathbb{R} \times \mathbb{R} \times (0, \infty) \times \mathbb{R}$, Problem (1.1)–(1.3) *has a unique solution satisfying* (1.7). *In addition, the solution can be extended to $r \in (0, \infty)$ and it depends continuously in $(c, d, m, p) \in \mathbb{R} \times \mathbb{R} \times (0, \infty) \times \mathbb{R}$.*

The idea of the proof of the existence part of Theorem 1.1 is to find appropriate values of c and d such that the corresponding solution given in Lemma 1.3 satisfies (1.4) and (1.5).

The proof of the nonexistence part of Theorem 1.1 is based on the following asymptotic behavior, as $r \to \infty$, of the solution of (1.1)–(1.5):

LEMMA 1.4. *Assume that* $m > 0$, $p > 0$, *and that* (u, v) *is a solution of Problem* (1.1)–(1.3) *satisfying*

$$u' > 0, \qquad v' > 0, \qquad u < \pi, \qquad v < 1 \qquad \text{for} \quad r \in (0, \infty). \tag{1.8}$$

Then for some $C > 0$ *and* $v_\infty \in (0, 1)$, (u, v) *has the following expansion, as* $r \to \infty$,

$$u(r) = \pi - Ce^{-\sqrt{p}r}[r^{-1/2} + O(r^{-3/2})], \tag{1.9}$$

$$u'(r) = Ce^{-\sqrt{p}r}[-\sqrt{p}r^{-1/2} + O(r^{-3/2})], \tag{1.10}$$

$$v(r) = v_\infty - \frac{(1 - v_\infty)C^2}{2p} e^{-2\sqrt{p}r}[r^{-1} + O(r^{-2})], \tag{1.11}$$

$$v'(r) = \frac{(1 - v_\infty)C^2}{\sqrt{p}} e^{-2\sqrt{p}r}[r^{-1} + O(r^{-2})]. \tag{1.12}$$

Lemmas 1.3 and 1.4 were first derived in [1], where p in the denominator on the right-hand side of (1.11) was missing. Here for completeness and reader's convenience, we shall provide their proofs in Section 5.

We shall prove the nonexistence part of Theorem 1.1 in the next section and the existence part in Sections 3 and 4. The proofs of some technical lemmas used in the paper are left to Section 5.

Throughout the paper, we shall always assume that $m > 0$ is fixed.

2. NONEXISTENCE OF SOLUTIONS WHEN $p \geqslant 4m$

The numerical experiment in [1] only deals with the existence of solutions to the BVP (1.1)–(1.5) for small $p > 0$. Here in this section we consider the opposite case; namely prove the nonexistence of solutions of (1.1)–(1.5) when $p \geqslant 4m$.

THEOREM 2.1. *If* $p \geqslant 4m$, *then Problem* (1.1)–(1.5) *has no solution.*

Proof. Suppose that (1.1)–(1.5) has a solution (u, v). We shall show that $p < 4m$.

Multiplying (1.1) by $2r^2u'$ and then integrating from 0 to r yields

$$r^2(u')^2 = [pr^2 + m^2(v - 1)^2] \sin^2 u + H(r), \tag{2.1}$$

where

$$H(r) = \int_0^r 2t \sin^2 u \cdot h(t)\, dt, \qquad h(r) = m^2(1 - v)\, v'/r - p.$$

Since $v' > 0$ and $(v'/r)' = -2(1-v)\sin^2 u/r < 0$, it follows that h is strictly decreasing in $(0, \infty)$. Also it follows from (2.1) and the asymptotic behavior of (u, v) in (1.9)–(1.12) that $\lim_{r\to\infty} H(r) = 0$. Therefore $h(0) > 0$ since otherwise $h(r) < 0$ in $(0, \infty)$, which implies that H is decreasing in $(0, \infty)$ and therefore $H(r) < H(1) < 0$ for all $r > 1$, which contradicts to $H(\infty) = 0$. From $h(0) > 0$ we immediately obtain

$$p < m^2 a, \qquad \text{where} \quad a := \lim_{r\to 0+} \frac{v'}{r}. \tag{2.2}$$

Also, h decreasing and $\lim_{r\to\infty} h(r) = -p$ implies that h has a unique zero, say at $r^* \in (0, \infty)$. Hence $H(r)$ is increasing in $[0, r^*]$ and is decreasing in $[r^*, \infty)$. As $H(0) = H(\infty) = 0$, we conclude that $H(r) > 0$ in $(0, \infty)$. Therefore from (2.1) we obtain

$$ru' \geqslant \sqrt{pr^2 + m^2(1-v)^2}\,\sin u \geqslant m(1-v)\sin u. \tag{2.3}$$

Now multiplying (1.2) by $1/r$ and integrating over $[0, \infty)$ yields

$$\begin{aligned} a &= 2\int_0^\infty \frac{(1\text{-}v)\sin^2 u}{r}\,dr = 2\int_0^\infty \frac{(1\text{-}v)\sin^2 u}{ru'}\,u'\,dr \\ &< \int_0^\infty \frac{2(1\text{-}v)\sin^2 u}{m(1\text{-}v)\sin u}\,u'\,dr = \frac{2}{m}\int_0^\pi \sin u\,du = \frac{4}{m}. \end{aligned} \tag{2.4}$$

Finally combining (2.2) with (2.4) we obtain that

$$p < m^2 a < m^2\,\frac{4}{m} = 4m.$$

This completes the proof. ∎

3. EXISTENCE OF SOLUTIONS WHEN $m > 0$

To show the existence of a solution to our BVP, we shall use a 2-dimensional topological shooting argument. Instead of using this argument directly to the solutions of (1.1), (1.2), and (1.7), we would rather use it to the solutions of the problem after the following scaling:

$$r = \varepsilon t, \qquad \text{where} \quad c\varepsilon^m = 2. \tag{3.1}$$

Then equations (1.1)–(1.5), and (1.7) become

$$\ddot{u}+\frac{1}{t}\,\dot{u}-\left(\frac{m^2(1-v)^2}{t^2}+p\varepsilon^2\right)\sin u\cos u=0 \qquad \text{for} \quad t>0, \tag{3.2}$$

$$\ddot{v}-\frac{1}{t}\,\dot{v}+2\varepsilon^2(1-v)\sin^2 u=0 \qquad \text{for} \quad t>0, \tag{3.3}$$

$$u(t)=2t^m+o(t^m) \qquad \text{as} \quad t\to 0, \tag{3.4}$$

$$v(t)=\frac{1}{2}\,\lambda t^2+o(t^2) \qquad \text{as} \quad t\to 0 \qquad (\lambda=2d\varepsilon^2), \tag{3.5}$$

$$\dot{u}>0, \qquad u(\infty)=\pi, \qquad \dot{v}>0, \qquad v(\infty)\leqslant 1, \tag{3.6}$$

where $\cdot = d/dt$. Note that the proof of the existence of a solution to (1.1)–(1.5) is equivalent to find (λ, ε) such that the unique solution of (3.2)–(3.5) (whose existence is given by Lemma 1.3) satisfies (3.6). In what follows, we always refer (u, v) to the solution of (3.2)–(3.5), where the dependence of (u, v) on p, ε, and λ is suppressed.

Now we outline the shooting argument. First for any given $p\geqslant 0$ and $\varepsilon>0$, we show in Lemma 3.2 that for sufficiently large positive λ, v reaches 1 before $\dot{v}$ reaches 0, and in Lemma 3.3 that for sufficiently small λ, $\dot{v}$ reaches 0 before v reaches 1. As v can not reach 1 at the same time as $\dot{v}$ reaches 0, the sets of λ with the above properties are open and disjoint. Hence their compliment $\mathbf{C}(p, \varepsilon)$ is a nonempty closed set of $(0, \infty)$ and has the property that for $\lambda\in\mathbf{C}(p, \varepsilon)$ the solution to (3.2)–(3.5) satisfies $\dot{v}>0$ and $0<v<1$ for all $t\geqslant 0$. Then we restrict λ to the set $\mathbf{C}(p, \varepsilon)$ in Lemmas 3.5 and 3.6 and show that for any fixed $p>0$ when ε is sufficiently large, $\dot{u}$ reaches 0 before u reaches π, and when $\varepsilon>0$ is fixed and $p\geqslant 0$ is sufficiently small, u reaches π before $\dot{u}$ reaches 0. Finally, by a topological argument, for any sufficiently small positive p there exists at least one (ε, λ) with $\lambda\in\mathbf{C}(p, \varepsilon)$ such that the solution to (3.2)–(3.5) satisfies $\dot{u}>0$ and $u\in(0, \pi)$ for all $t>0$, which, together with the property for v (since $\lambda\in\mathbf{C}(p, \varepsilon)$), yields (3.6), thereby establishing the existence of a solution to (3.2)–(3.6).

We begin our existence proof with some properties of u and v.

LEMMA 3.1. *Let $p\geqslant 0$, $\varepsilon>0$, and $\lambda>0$ be arbitrarily given, and let (u, v) be the solution of* (3.2)–(3.5).

(1) *There is a $t_1>0$ such that $\dot{u}>0$ in $(0, t_1]$ and $u(t_1)=\pi/2$.*

(2) *Let $[0, T]\subseteq[0, t_1]$ be the maximal interval where $\dot{v}\geqslant 0$ and $v\in(0, 1]$. Then in $(0, T]$,*

$$m(1-v)\sin u < t\dot{u} < (m+\sqrt{p}\varepsilon t)\sin u, \tag{3.7}$$

$$t^m e^{-(m\lambda/4)\,t^2} < \tan\frac{u}{2} < t^m e^{\varepsilon\sqrt{p}t}, \tag{3.8}$$

$$\lambda - \frac{4\varepsilon^2}{m}\sin^2\frac{u}{2} < \frac{\dot{v}}{t} < \lambda - \frac{4\varepsilon^2(1-\frac{1}{2}\lambda t^2)}{m+\varepsilon\sqrt{p}t}\sin^2\frac{u}{2}. \tag{3.9}$$

Proof. (1) Write (3.2) as

$$(t\dot{u})\cdot = t\left[\frac{m^2(v-1)^2}{t^2} + p\varepsilon^2\right]\sin u\cos u. \tag{3.10}$$

We see that in the interval where $u \in (0, \pi/2)$, $t\dot{u}$ is increasing and so $\dot{u} > 0$. Now if t_1 does not exist, then $u \in (0, \pi/2)$ in $(0, \infty)$, and $t\dot{u} > \dot{u}(1)$ for all $t > 1$, which, after integration, yields $u(t) \geqslant u(1) + \dot{u}(1)\ln t$ for all $t \geqslant 1$, contradicting $u \in (0, \pi/2)$. Therefore t_1 exists.

(2) Multiplying (3.10) by $t\dot{u}$ and integrating over $[0, t]$ we get

$$\frac{(t\dot{u})^2}{2} = \int_0^t \dot{u}\sin u\cos u[m^2(1-v)^2 + p\varepsilon^2 s^2]\,ds. \tag{3.11}$$

Then (3.7) follows at once from the fact that $\int_0^t \dot{u}\sin u\cos u\,ds = \frac{1}{2}\sin^2 u$, and

$$m^2(1-v(t))^2 < m^2(1-v(s))^2 + p\varepsilon^2 s^2 < (m+\varepsilon\sqrt{p}t)^2$$

for all $s \in (0, t)$, $t \in (0, T]$.

To show (3.8), we need a preliminary estimate on v. Multiplying (3.3) by $1/t$ and then integrating on $[0, t]$ we get

$$\frac{\dot{v}}{t} = \lambda - 2\varepsilon^2\int_0^t \frac{(1-v)\sin^2 u}{s}\,ds. \tag{3.12}$$

Consequently,

$$\dot{v}(t) < \lambda t, \qquad v(t) < \tfrac{1}{2}\lambda t^2 \qquad \forall t \in (0, T]. \tag{3.13}$$

Now we prove (3.8). Dividing (3.7) by $t\sin u$, integrating the resulting inequalities over $[t_0, t]$, $0 < t_0 \ll 1$, and then using the fact that $\int_{u(t_0)}^u 1/\sin x\,dx = \ln(\tan(u/2)/\tan(u(t_0)/2))$ and $v < \lambda t^2/2$, we get, $\forall t \in [t_0, T]$,

$$\tan\frac{u(t_0)}{2}\left[\left(\frac{t}{t_0}\right)^m e^{-(m\lambda/4)(t^2-t_0^2)}\right] < \tan\frac{u}{2} < \tan\frac{u(t_0)}{2}\left[\left(\frac{t}{t_0}\right)^m e^{\varepsilon\sqrt{p}(t-t_0)}\right].$$

Sending $t_0 \searrow 0$, we obtain (3.8) with the strict inequality replaced by "$\leqslant$". Then substituting the obtained estimate for $\tan u(t_0)/2$ into the above strict inequality, we get (3.8) with strict inequality.

Finally, we prove (3.9). Substituting $(1-v)\sin u$ in (3.12) by its dominator $s\dot{u}/m$ we obtain, $\forall t \in (0, T]$,

$$\frac{\dot{v}}{t} > \lambda - 2\varepsilon^2 \int_0^t \frac{\dot{u}\sin u}{m}\, ds = \lambda - \frac{4\varepsilon^2}{m} \sin^2 \frac{u}{2}.$$

Similarly, replacing $(1-v)$ and $\sin u$ in (3.12) by $1-\frac{1}{2}\lambda s^2$ and $s\dot{u}(s)/(m+\varepsilon\sqrt{ps})$ respectively, we obtain, $\forall t \in (0, T]$,

$$\frac{\dot{v}}{t} < \lambda - 2\varepsilon^2 \int_0^t \frac{(1-\frac{1}{2}\lambda s^2)\,\dot{u}\sin u}{m+\varepsilon\sqrt{ps}}\, ds \leqslant \lambda - \frac{4\varepsilon^2(1-\frac{1}{2}\lambda t^2)}{m+\varepsilon\sqrt{pt}} \sin^2 \frac{u}{2}$$

since $(1-\frac{1}{2}\lambda t^2)/(m+\varepsilon\sqrt{pt})$ is decreasing. This completes the proof of the lemma. ∎

We now consider the case when λ is large.

LEMMA 3.2. *For any given $p \geqslant 0$ and $\varepsilon > 0$, if*

$$\lambda \geqslant \lambda_1(p, \varepsilon) := \max\left\{\left(2p + \frac{2}{m}\right)\varepsilon^2, \frac{2\varepsilon^2}{m} + 2e^{2/m}\right\}, \tag{3.14}$$

then there exists $T_1 \in (0, t_1)$ such that $v(T_1) = 1$ and $\dot{v} > 0$ in $(0, T_1]$.

Proof. Let $(0, T_1)$ be the maximal interval in $(0, t_1]$ such that $\dot{v} > 0$ and $v < 1$. Then since $u \leqslant \pi/2$ in $[0, t_1]$, from the first inequality in (3.9), we have, for all $t \in (0, T_1]$,

$$\dot{v}(t) > \left(\lambda - \frac{2\varepsilon^2}{m}\right) t, \qquad v(t) > \left(\lambda - \frac{2\varepsilon^2}{m}\right) \frac{t^2}{2}.$$

Since $\lambda > 2\varepsilon^2/m$, we see that $\dot{v} > 0$ in $(0, T_1]$. Hence, to finish the proof, we need only show that $T_1 < t_1$. It suffices to show that $(\lambda - 2\varepsilon^2/m)(t_1^2/2) \geqslant 1$, i.e., $t_1 \geqslant \sqrt{2/(\lambda - 2\varepsilon^2/m)} =: \bar{t}$. In fact, if $T_1 = t_1 < \bar{t}$, then by the second inequality of (3.8) we have,

$$1 = \tan\frac{u(t_1)}{2} \leqslant t_1^m e^{\varepsilon\sqrt{pt_1}} < \bar{t}^m e^{\varepsilon\sqrt{p\bar{t}}} \leqslant 1 \qquad \text{(by (3.14))},$$

which is impossible. This proves the lemma. ∎

Next, we study the case when $\lambda > 0$ is small.

LEMMA 3.3. *For any given $p \geqslant 0$ and $\varepsilon > 0$, if*

$$0 < \lambda \leqslant \lambda_2(p, \varepsilon) := \min \left\{ \frac{2}{e}, \frac{2\varepsilon^2}{m + \varepsilon\sqrt{ep} + e\varepsilon^2} \right\}, \tag{3.15}$$

then there exists $T_2 \in (0, t_1)$ such that $\dot{v}(T_2) = 0$ and $v \in (0, 1)$ in $(0, T_2]$.

Proof. Let $(0, T_2)$ be the maximal interval in $(0, t_1]$ where $\dot{v} > 0$ and $v < 1$.

Since $\lambda \leqslant 2/e$, the definition of T_2 and the first inequality of (3.8) implies

$$1 \geqslant \tan \frac{u(T_2)}{2} \geqslant t^m e^{-(m\lambda/4)t^2} > t^m e^{-(m/2e)t^2} \qquad \forall t \in (0, T_2],$$

so that $T_2 \leqslant \sqrt{e}$, and from (3.13), $v(T_2) < \lambda T_2^2/2 < 1$. Hence, to finish the proof, it suffices to show that $T_2 < t_1$.

In fact, if $T_2 = t_1$, then, from the second inequality in (3.9),

$$\begin{aligned}
\frac{\dot{v}(T_2)}{T_2} &< \lambda - \frac{1 - \frac{1}{2}\lambda T_2^2}{m + \varepsilon\sqrt{pT_2}}\, 2\varepsilon^2 \\
&\leqslant \lambda - \frac{1 - \frac{1}{2}\lambda e}{m + \varepsilon\sqrt{ep}}\, 2\varepsilon^2 \qquad \left(\text{since } \frac{1 - \frac{1}{2}\lambda t^2}{m + \varepsilon\sqrt{pt}} \text{ is decreasing}\right) \\
&= \lambda \left(1 + \frac{e\varepsilon^2}{(m + \varepsilon\sqrt{pe})}\right) - \frac{2\varepsilon^2}{m + \varepsilon\sqrt{pe}} \leqslant 0 \qquad (\text{by (3.15)}),
\end{aligned}$$

which contradicts the definition of T_2. Thus $T_2 < t_1$ and the assertion of the lemma follows. ∎

Now, for every $\varepsilon > 0$ and $p \geqslant 0$, we define

$$\begin{aligned}
\Lambda_1(p, \varepsilon) &= \{\lambda > 0 : \exists T_1 > 0 \ni v(T_1) = 1 \ \&\ \dot{v} > 0 \text{ in } (0, T_1)\}, \\
\Lambda_2(p, \varepsilon) &= \{\lambda > 0 : \exists T_2 > 0 \ni \dot{v}(T_2) = 0 \ \&\ v < 1 \text{ in } (0, T_2)\}, \\
\mathbf{C}(p, \varepsilon) &= (0, \infty) \setminus (\Lambda_1(p, \varepsilon) \cup \Lambda_2(p, \varepsilon)).
\end{aligned}$$

LEMMA 3.4. *Let $p \geqslant 0$ and $\varepsilon > 0$ be given and let $\Lambda_1(p, \varepsilon)$, $\Lambda_2(p, \varepsilon)$, and $\mathbf{C}(p, \varepsilon)$ be defined as above. Then the following holds*:

(a) $\Lambda_1(p, \varepsilon) \supset [\lambda_1(p, \varepsilon), \infty)$, $\Lambda_2(p, \varepsilon) \supset (0, \lambda_2(p, \varepsilon)]$, *and* $\mathbf{C}(p, \varepsilon) \subset (\lambda_2(p, \varepsilon), \lambda_1(p, \varepsilon))$.

(b) *Both the set* $\bigcup_{p\geqslant 0,\, \varepsilon>0} \{(p, \varepsilon)\} \times \Lambda_1(p, \varepsilon)$ *and the set* $\bigcup_{p\geqslant 0,\, \varepsilon>0} \{(p, \varepsilon)\} \times \Lambda_2(p, \varepsilon)$ *are open and disjoint in* $(p, \varepsilon, \lambda) \in [0, \infty) \times (0, \infty) \times (0, \infty)$. *Consequently, for every* $p_1 > 0$ *and* $\varepsilon_1 > 0$, *the set* $\mathbf{C}(p_1, \varepsilon_1)$ *is non-empty and the set* $\bigcup_{p\in[0, p_1]} \{p\} \times \mathbf{C}(p, \varepsilon_1)$ *is a compact set in* $[0, \infty) \times (0, \infty)$.

(c) *For every* $p \geqslant 0, \varepsilon > 0$ *and* $\lambda \in \mathbf{C}(p, \varepsilon)$, *the solution* (u, v) *to* (3.2)–(3.5) *satisfies*

$$\dot{v} > 0, \qquad 0 < v < 1 \qquad \textit{for all} \quad t > 0.$$

Proof. (a) The first assertion follows immediately from Lemmas 3.2 and 3.3 and the definition of Λ_1, Λ_2 ad $\mathbf{C}$.

(b) Since $v(t_0) = 1$ and $\dot{v}(t_0) = 0$ for some $t_0 > 0$ implies $v \equiv 1$, it then follows that Λ_1 and Λ_2 are disjoint.

As $\lambda \in \Lambda_1(p, \varepsilon)$ implies that $\dot{v}(T_1) \neq 0$, the assertion that $\bigcup_{p\geqslant 0,\, \varepsilon>0} \{(p, \varepsilon)\} \times \Lambda_1(p, \varepsilon)$ is open then follows from the continuous dependence of solutions of (3.2)–(3.5) with respect to the parameter $(p, \varepsilon, \lambda)$.

Similarly, to show that $\bigcup_{p\geqslant 0,\, \varepsilon>0} \{(p, \varepsilon)\} \times \Lambda_2(p, \varepsilon)$ is open, it suffices to show that either $\ddot{v}(T_2) \neq 0$ or $\ddot{v}(T_2) = \dddot{v}(T_2) = 0$ and $\ddddot{v}(T_2) \neq 0$ when $\lambda \in \Lambda_2(p, \varepsilon)$. In fact, if $\ddot{v}(T_2) = 0$, then the differential equation for v implies that $u(T_2) = k\pi$ for some integer k and that $\dot{u}(T_2) \neq 0$ (else $u \equiv k\pi$), so that differentiating the equation for v, we obtain $\dddot{v}(T_2) = 0$ and $\ddddot{v}(T_2) \neq 0$. Thus, $\bigcup_{p\geqslant 0,\, \varepsilon>0} \{(p, \varepsilon)\} \times \Lambda_2(p, \varepsilon)$ is open.

The assertion for the set $\mathbf{C}$ is a consequence of (a) and the fact that $(0, \infty)$ can not be written as the union of two non-empty disjoint open sets.

(c) Since initially $\dot{v} > 0$ and $v \in (0, 1)$, and a violation of one of these two properties will imply that λ is in $\Lambda_1(p, \varepsilon) \cup \Lambda_2(p, \varepsilon)$, the assertion (c) thus follows. ∎

Now we shall restrict our attention to the case when $\lambda \in \mathbf{C}(p, \varepsilon)$. We shall show that for any fixed $p > 0$, $\dot{u}$ reaches zero before u reaches π for large ε, whereas for any fixed ε with p sufficiently small, u reaches π before $\dot{u}$ reaches zero.

LEMMA 3.5. *For every* $p > 0$, *there exists an* $\bar{\varepsilon} := \bar{\varepsilon}(p) > 0$ *such that if* $\varepsilon \geqslant \bar{\varepsilon}$, *then for any* $\lambda \in \mathbf{C}(p, \varepsilon)$, *the solution of* (3.2)–(3.5) *satisfies, for some* $T_3 > 0$,

$$\dot{u} > 0 \quad \textit{and} \quad u < \pi \ \textit{in} \ (0, T_3), \qquad \textit{and} \qquad \dot{u}(T_3) = 0. \tag{3.16}$$

Proof. Let $p > 0$ be fixed. We first show that there exists an $\bar{\varepsilon} := \bar{\varepsilon}(p) > 0$ such that if $\varepsilon > \bar{\varepsilon}$ and $\lambda \in \mathbf{C}(p, \varepsilon)$ then

$$\dot{u}^2(t_1) < p\varepsilon^2, \tag{3.17}$$

where $t_1 > 0$ is as in Lemma 3.1.

Taking $t=t_1$ in (3.11) we have

$$\begin{aligned}\frac{1}{2}(t_1\dot u(t_1))^2 &= \int_0^{t_1} \dot u \sin u \cos u\,[m^2(v-1)^2+p\varepsilon^2 t^2]\,dt\\ &\leqslant \int_0^{t_1} \dot u \sin u \cos u\,[m^2+p\varepsilon^2 t^2]\,dt\\ &= \frac{m^2}{2}+\frac{p\varepsilon^2 t_1^2}{2}-p\varepsilon^2\int_0^{t_1} t\sin^2 u\,dt. \end{aligned}\tag{3.18}$$

Therefore to show (3.17), it suffices to show that the last term is bigger than $m^2/2$ when $\varepsilon \gg 1$. Let $t_2\in(0,t_1)$ be the time such that $u(t_2)=\pi/4$. Then (3.8) implies $t_2^m e^{\varepsilon\sqrt{p}t_2}\geqslant \tan\pi/8$, so that

$$t_2 \geqslant \frac{1}{\varepsilon\sqrt{p}}\ln|\ln(\varepsilon\sqrt{p})| \qquad \text{provided that } \varepsilon \gg 1.$$

It then follows from the second inequality of (3.7) that

$$\begin{aligned}p\varepsilon^2\int_0^{t_1} t\sin^2 u\,dt &\geqslant p\varepsilon^2\int_{t_2}^{t_1}\frac{t^2\dot u\sin u}{m+\sqrt{p}\varepsilon t}\,dt\\ &\geqslant \frac{p\varepsilon^2 t_2^2}{m+\sqrt{p}\varepsilon t_2}\int_{\pi/4}^{\pi/2}\sin u\,du, \quad \left(\text{since } \frac{t^2}{m+\sqrt{p}\varepsilon t} \text{ is increasing}\right)\\ &= \frac{p\varepsilon^2\,t_2^2}{\sqrt{2}(m+\sqrt{p}\varepsilon t_2)} \geqslant \frac{1}{2}\cdot \ln|\ln(\varepsilon\sqrt{p})|,\end{aligned}$$

provided that $\varepsilon \gg 1$. Substituting this estimate into (3.18) we then conclude that there exists $\bar\varepsilon := \bar\varepsilon(p)$ such that (3.17) holds for all $\varepsilon > \bar\varepsilon$ and $\lambda\in \mathbf{C}(p,\varepsilon)$.

To finish the proof, it suffices to show that (3.17) implies (3.16). For this purpose, we define an "energy" function

$$V(t)=\frac{1}{2}\dot u^2+\frac{1}{2}\cos^2 u\left[\frac{m^2(v-1)^2}{t^2}+p\varepsilon^2\right].$$

Then

$$\dot V(t) = -m^2\cos^2 u\,\frac{(1-v)\,\dot v t+(1-v)^2}{t^3}\leqslant 0,$$

so that V is decreasing for all $t>0$. Hence $V(t)<V(t_1)=\dot u^2(t_1)/2<p\varepsilon^2/2$ for all $t>t_1$ by (3.17). It then yields that $u(t)<\pi$ for all $t>t_1$ since $u(t)=\pi$ at some $t>t_1$ would implies that $V(t)>p\varepsilon^2/2$ by the definition of V. Now if T_3 does not exist, then $\dot{u}(t) > 0$ for all $t > 0$, which implies that $\dot{u}(\infty) = \ddot{u}(\infty) = 0$ and $u(\infty) = \pi$. But this would imply that $V(\infty) = p\varepsilon^2/2$, contradicting the fact that $V(\infty) \leqslant V(t_1) < p\varepsilon^2/2$. This completes the proof of the lemma. ∎

In order to complete our shooting argument for the existence, we would like to show that for any sufficiently small $\varepsilon > 0$ and $\lambda \in \mathbf{C}(p, \varepsilon)$, the solution of (3.2)–(3.5) satisfies that u reaches π before $\dot{u}$ reaches 0. However, from our nonexistence theorem this cannot be true for $p \geqslant 4m$ since otherwise our shooting argument (to be presented at the end of this section) would yield a solution. Hence, we have to restrict our attention to small $p > 0$.

LEMMA 3.6. *For every given $\varepsilon_1 > 0$, there exists a $\bar{p} = \bar{p}(\varepsilon_1) > 0$ such that for every $p \in [0, \bar{p}(\varepsilon_1)]$ and $\lambda \in \mathbf{C}(p, \varepsilon_1)$, the solution to (3.2)–(3.5) satisfies, for some $T_4 > 0$,*

$$u(T_4) = \pi \qquad \textit{and} \qquad \dot{u} > 0 \qquad \textit{on} \quad (0, T_4). \tag{3.19}$$

Proof. The idea of the proof is to show that (3.19) is true for $p = 0$, and then use a continuation argument for small p. For the convenience of our presentation, we use a contradiction argument.

Suppose that the lemma is not true. Then there exists a sequence $\{(p_n, \lambda_n)\}_{n=1}^{\infty}$ such that $p_n > 0$ and $\lambda_n \in \mathbf{C}(p_n, \varepsilon_1)$ for all n, that $\lim_{n \to \infty} p_n = 0$, and that the solution (u_n, v_n) of (3.2)–(3.5) with $(p, \varepsilon, \lambda) = (p_n, \varepsilon_1, \lambda_n)$ satisfies

$$\text{either} \quad \dot{u}_n(t_n) = 0 \quad \text{and} \quad u_n < \pi \quad \text{in } (0, t_n) \text{ for some } t_n > 0, \tag{3.20}$$

$$\text{or} \quad \dot{u}_n > 0, \quad u_n < \pi \quad \text{in} \quad (0, \infty). \tag{3.21}$$

We now show that none of them are possible if n is large enough.

Observe that Lemma 3.4 (b) implies $\{(p_n, \lambda_n)\}_{n=1}^{\infty}$ has a subsequence which converges to some $(0, \lambda_0)$ with $\lambda_0 \in \mathbf{C}(0, \varepsilon_1)$. Now let (u, v) be the solution of (3.2)–(3.5) with $(p, \varepsilon, \lambda) = (0, \varepsilon_1, \lambda_0)$. Since $\lambda_0 \in \mathbf{C}(0, \varepsilon_1)$, $v \in (0, 1)$ and $\dot{v} > 0$ for all $t > 0$. It then follows by integration by parts for the right-hand side of (3.11) that

$$\begin{aligned} \frac{1}{2}(t\dot{u})^2 &= \frac{m^2}{2}(v-1)^2 \sin^2 u + m^2 \int_0^t (1-v)\dot{v} \sin^2 u \, ds \\ &> m^2 \int_0^{t_1} (1-v)\dot{v} \sin^2 u \, ds =: \frac{b^2}{2} \qquad \text{for all} \quad t > t_1, \end{aligned}$$

which implies that $\dot{u}(t) > |b|/t$ for all $t > t_1$ and therefore $u(t) > \pi/2 + |b| \ln t$ for all $t > t_1$. Hence

$$u(t_1 + e^{3/|b|}) \geqslant \pi + 1 \qquad \text{and} \qquad \dot{u} > 0 \qquad \text{in} \quad (0, t_1 + e^{3/|b|}], \tag{3.22}$$

which implies, by the continuity of solutions in (p, λ) again, that (3.22) holds for u_n with n sufficiently large, contradicting both (3.20) and (3.21). The assertion of the lemma thus follows. ∎

Now we are ready to prove the following existence theorem.

THEOREM 3.7. *For every given* $m > 0$, *there exists* $p^* = p^*(m) \geqslant \sup_{\varepsilon_1 > 0} \bar{p}(\varepsilon_1)$, *where* $\bar{p}(\varepsilon_1)$ *is as in Lemma* 3.6, *such that for any* $p \in (0, p^*)$, *the solution of* (3.2)–(3.5), *for some* $\varepsilon > 0$ *and* $\lambda \in \mathbf{C}(p, \varepsilon)$, *satisfies* (3.6). *Consequently*, *Problem* (1.1)–(1.5) *admits at least a solution.*

Proof. We shall show that for any given $\varepsilon_1 > 0$, Problem (3.2)–(3.6) has a solution if $p \in (0, \bar{p}(\varepsilon_1))$. Taking the best ε_1 then yields that there exists $p^* \geqslant \sup_{\varepsilon_1 > 0} \bar{p}(\varepsilon_1)$ such that Problem (3.2)–(3.6) has a solution if $p \in (0, p^*)$.

Now let $\varepsilon_1 > 0$ be any fixed number and $p \in (0, \bar{p}(\varepsilon_1))$. We want to show that (3.2)–(3.6) has a solution. To do this, we set

$$\varepsilon_2 := \max\{\bar{\varepsilon}(p), 2\varepsilon_1\}, \qquad \bar{\lambda}_1 := \max_{\varepsilon \in [\varepsilon_1, \varepsilon_2]} \lambda_1(p, \varepsilon) < \infty,$$

$$\bar{\lambda}_2 := \min_{\varepsilon \in [\varepsilon_1, \varepsilon_2]} \lambda_2(p, \varepsilon) > 0,$$

where $\bar{\varepsilon}(p)$ is defined in Lemma 3.5 and λ_1 and λ_2 are defined in Lemmas 3.2 and 3.3.

We define

$$\mathscr{S}_1 = \{(\varepsilon,\lambda) \in [\varepsilon_1,\varepsilon_2] \times [\bar{\lambda}_2,\bar{\lambda}_1] : \lambda \in \Lambda_1(p,\varepsilon)\},$$

$$\mathscr{S}_2 = \{(\varepsilon,\lambda) \in [\varepsilon_1,\varepsilon_2] \times [\bar{\lambda}_2,\bar{\lambda}_1] : \lambda \in \Lambda_2(p,\varepsilon)\}.$$

Then Lemma 3.4 implies that $\mathscr{S}_1 \supset [\varepsilon_1, \varepsilon_2] \times \{\bar{\lambda}_1\}$ and $\mathscr{S}_2 \supset [\varepsilon_1, \varepsilon_2] \times \{\bar{\lambda}_2\}$. Also, both sets are disjoint and open (with respect to $[\varepsilon_1, \varepsilon_2] \times [\bar{\lambda}_2, \bar{\lambda}_1]$). Therefore by a topological lemma (see [3], for example) there exists a closed and connected set $\mathscr{C} \subset [\varepsilon_1, \varepsilon_2] \times [\bar{\lambda}_2, \bar{\lambda}_1] \setminus (\mathscr{S}_1 \cup \mathscr{S}_2) = \bigcup_{\varepsilon \in [\varepsilon_1, \varepsilon_2]} \{\varepsilon\} \times \mathbf{C}(p, \varepsilon)$, connecting the lines $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$. Observe that, for any $(\varepsilon, \lambda) \in \mathscr{C}$, $\dot{v} > 0$ and $v \in (0, 1)$ in $(0, \infty)$.

Next we define another two sets on $\mathscr{C}$:

$$\mathscr{A} = \{(\varepsilon, \lambda) \in \mathscr{C} : \exists T_4 > 0 \ni u(T_4) = \pi, \ \dot{u} > 0 \text{ on } (0, T_4)\},$$

$$\mathscr{B} = \{(\varepsilon,\lambda) \in \mathscr{C} : \exists T_3 > 0 \ni \dot{u}(T_3) = 0, \ u < \pi \text{ in } (0,T_3)\}.$$

Then Lemma 3.5 yields $\mathscr{B} \supset (\{\varepsilon_2\} \times [\bar{\lambda}_2, \bar{\lambda}_1)]) \cap \mathscr{C}$, and Lemma 3.6 yields $\mathscr{A} \supset (\{\varepsilon_1\} \times [\bar{\lambda}_2, \bar{\lambda}_1]) \cap \mathscr{C}$. As $\dot{u}=0$ and $u=\pi$ at a same point $t_0>0$ implies that $u \equiv \pi$ for all $t \geqslant 0$, $\mathscr{A}$ and $\mathscr{B}$ are disjoint. Also, since $\dot{u}(T_4) \neq 0$ if $(\varepsilon, \lambda) \in \mathscr{A}$ and $\ddot{u}(T_3) \neq 0$ if $(\varepsilon, \lambda) \in \mathscr{B}$, the implicit function theorem implies that $\mathscr{A}$ and $\mathscr{B}$ are relatively open in $\mathscr{C}$. Hence, since $\mathscr{C}$ is closed and connected, there is a $(\varepsilon^*, \lambda^*) \in \mathscr{C} \setminus (\mathscr{A} \cup \mathscr{B})$. Now by the definition of $\mathscr{A}$ and $\mathscr{B}$, the solution of (3.2)–(3.5) with $(\varepsilon, \lambda) = (\varepsilon^*, \lambda^*)$ satisfies $\dot{u}>0$ and $u \in (0, \pi)$ in $(0, \infty)$, which implies $u(\infty) = \pi$ by equation (3.2). Therefore the solution satisfies (3.6). This completes the proof of the theorem. ∎

Remark 3.8. Since ε_1 in the proof can be arbitrarily large, we can take ε^* such that $\lim_{p \searrow 0} \varepsilon^* = \infty$.

4. A LOWER BOUND FOR $p^*(m)$ WHEN $m>1$

Theorem 3 did not provide any explicit estimate for the lower bound of $p^*(m)$ though it states that $p^*(m)$ is bounded from below by $\sup_{\varepsilon_1>0} \bar{p}(\varepsilon_1)$. The purpose of this section is to give an explicit lower bound for $p^*(m)$ when $m>1$, by estimating $\limsup_{\varepsilon \searrow 0} \bar{p}(\varepsilon)$. In the sequel, $m>1$ is fixed and all m dependence are suppressed.

We need the following lemma.

LEMMA 4.1. *Let $p \geqslant 0$ be given. Then*

$$\limsup_{\varepsilon \searrow 0} \sup_{\lambda \in \mathbf{C}(p, \varepsilon)} \frac{\lambda}{\varepsilon^2 (\ln \varepsilon)^2} \leqslant 1. \tag{4.1}$$

Also, if $0<\varepsilon \ll 1$ and $\lambda \in \mathbf{C}(p, \varepsilon)$, then for all $t \in (0, t_1]$, where t_1 is as in Lemma 3.1,

$$t\dot{u} = m[1+O(\varepsilon)] \sin u, \qquad \tan \frac{u}{2} = t^m [1+O(\varepsilon)], \qquad t_1 = 1+O(\varepsilon), \tag{4.2}$$

$$v(t) = O(\varepsilon), \qquad \frac{\dot{v}}{t} = \lambda - \frac{4\varepsilon^2}{m} \sin^2 \frac{u}{2} + O(\varepsilon^3). \tag{4.3}$$

Furthermore,

$$\liminf_{\varepsilon \searrow 0} \inf_{\lambda \in \mathbf{C}(p, \varepsilon)} \frac{\lambda m}{4\varepsilon^2} \geqslant 1. \tag{4.4}$$

We again leave its proof to Section 5.

Now we estimate $\bar{p}(\varepsilon)$ in Lemma 3.6 for ε sufficiently small. We assume that $\varepsilon > 0$ is sufficiently small, and $\lambda \in \mathbf{C}(p, \varepsilon)$.

As $\dot{v} > 0$ and $v \in (0, 1)$ in $(0, \infty)$, for all $t > t_1$, $\int_{t_1}^{t} (1-v)^2\, \dot{u} \sin u \cos u\, ds \geqslant \frac{1}{2}(1-v(t_1))^2 [\sin^2 u(t) - 1]$. Using the identity $\int_0^{t_1} (1-v)^2\, \dot{u} \sin u \cos u\, ds = \frac{1}{2}(1-v(t_1))^2 + \int_0^{t_1} (1-v)\, \dot{v} \sin^2 u\, ds$, we then obtain

$$\int_0^t (1-v)^2\, \dot{u} \sin u \cos u\, ds \geqslant \tfrac{1}{2}(1-v(t_1))^2 \sin^2 u(t) + \int_0^{t_1} (1-v)\, \dot{v} \sin^2 u\, ds$$

for all $t > t_1$. Consequently, from (3.11),

$$\tfrac{1}{2} t^2 \dot{u}^2 > \tfrac{1}{2} m^2 (1-v(t_1))^2 \sin^2 u + \alpha - \beta(t), \tag{4.5}$$

where

$$\alpha = m^2 \int_0^{t_1} (1-v)\, \dot{v} \sin^2 u\, ds + p\varepsilon^2 \int_0^{t_1} s^2 \dot{u} \sin u \cos u\, ds$$

$$\beta(t) = -p\varepsilon^2 \int_{t_1}^{t} s^2 \dot{u} \sin u \cos u\, ds.$$

We define

$$T_4 = \sup\{t > t_1 : \beta < (1-\varepsilon)\alpha,\ \dot{u} > 0,\ u < \pi \text{ in } (t_1, t)\}.$$

Since $\alpha > 0$, $\beta(t_1) = 0$ and $\dot{u}(t_1) > 0$, $T_4 \in (t_1, \infty]$ is well-defined. In $[t_1, T_4)$, we have, from (4.5),

$$t\dot{u} \geqslant \sqrt{\hat{m}^2 \sin^2 u + 2\varepsilon\alpha} > \max\{\hat{m} \sin u, \sqrt{2\varepsilon\alpha}\}, \tag{4.6}$$

where $\hat{m} = m(1-v(t_1))$. This implies that $T_4 < \infty$. In addition, at T_4, either $\beta(T_4) = (1-\varepsilon)\alpha$ or $u(T_4) = \pi$. Now we show that if p is suitably small, the first alternative will not happen. To do this we estimate α and β.

First we estimate $\beta(t)$. Integrating (4.6) over $[t_1, t]$ with integrating factor $1/(t \sin u)$ we obtain, for $t \in [t_1, T_4]$, $t^{\hat{m}} \leqslant t_1^{\hat{m}} \tan u/2$. Therefore, from the definition of $\beta(t)$, for all $t \in [t_1, T_4]$,

$$\begin{aligned} \beta(t) &\leqslant p\varepsilon^2 \int_{t_1}^{t} \dot{u}\, \mathit{sin}\, u\, |\mathit{cos}\, u|\, t_1^2 \left[\tan \frac{u}{2}\right]^{2/\hat{m}} ds \\ &\leqslant p\varepsilon^2 [1 + O(\varepsilon)] \int_{\pi/2}^{\pi} \sin u\, |\cos u| \left[\tan \frac{u}{2}\right]^{2/m} du, \end{aligned}$$

since by Lemma 4.1, $t_1 = 1 + O(\varepsilon)$ and $\hat{m} = m(1-v(t_1)) = m[1 + O(\varepsilon)]$. Here we need the condition $m > 1$ to insure the convergence of the integral.

Next we estimate α. Using Lemma 4.1 we can calculate

$$
\begin{aligned}
&\int_0^{t_1} \sin^2 u(1\text{-}v)\dot{v}\,dt \\
&\quad = \int_0^{t_1} \frac{t^2\dot{u}\sin^2 u(1\text{-}v)}{t\dot{u}}\frac{\dot{v}}{t}\,dt \\
&\quad = \int_0^{t_1} \frac{(\tan(u/2)[1+O(\varepsilon)])^{2/m}\,\dot{u}\sin^2 u}{m\sin u[1+O(\varepsilon)]}\left[\lambda - \frac{4\varepsilon^2}{m}\sin^2\frac{u}{2} + O(\varepsilon^3)\right]dt \\
&\quad = \frac{1}{m^2}\int_0^{\pi/2}\left[\lambda m - 4\varepsilon^2\sin^2\frac{u}{2}\right]\sin u\left(\tan\frac{u}{2}\right)^{2/m} du + O(\varepsilon^3)
\end{aligned}
$$

and

$$
\int_0^{t_1} t^2\dot{u}\sin u\cos u\,dt = \int_0^{\pi/2}\sin u\cos u\left(\tan\frac{u}{2}\right)^{2/m} du + O(\varepsilon).
$$

It then follows from the definition of α that

$$
\begin{aligned}
\alpha = &\int_0^{\pi/2}\left[\lambda m - 4\varepsilon^2\sin^2\frac{u}{2}\right]\sin u\left(\tan\frac{u}{2}\right)^{2/m} du \\
&+ \varepsilon^2 p\int_0^{\pi/2}\sin u\cos u\left(\tan\frac{u}{2}\right)^{2/m} du + O(\varepsilon^3).
\end{aligned}
$$

Hence, $\beta(t) \leqslant (1-2\varepsilon)\alpha$ holds in $[t_1, T_4]$ provided that

$$
\begin{aligned}
p\int_{\pi/2}^{\pi}\sin u\ |\cos u|\left(\tan\frac{u}{2}\right)^{2/m} du < &\int_0^{\pi/2}\sin u\left[\frac{\lambda m}{\varepsilon^2} - 4\sin^2\frac{u}{2}\right]\left(\tan\frac{u}{2}\right)^{2/m} du \\
&+ p\int_0^{\pi/2}\sin u\cos u\left(\tan\frac{u}{2}\right)^{2/m} du - O(\varepsilon).
\end{aligned}
$$

Since from Lemma 4.1, $\lambda m/\varepsilon^2 \geqslant 4 - o(1)$, where $\lim_{\varepsilon\to 0} o(1) = 0$, we see that the last inequality is true provide that

$$
\begin{aligned}
p &\leqslant \frac{4\int_0^{\pi/2}\sin u\cos^2\frac{u}{2}\left(\tan\frac{u}{2}\right)^{2/m} du}{-\int_0^{\pi}\sin u\cos u\left(\tan\frac{u}{2}\right)^{2/m} du} - o(1) \\
&= 4m\,\frac{\int_0^{\pi/2}\sin u\cos^2\frac{u}{2}\left(\tan\frac{u}{2}\right)^{2/m} du}{\int_0^{\pi}\sin u\left(\tan\frac{u}{2}\right)^{2/m} du} - o(1). \qquad (4.7)
\end{aligned}
$$

Hence, if p satisfies the above inequality, then $u(T_4)=\pi$ and $\dot{u}>0$ in $(0, T_4]$. We then have the following lemma:

LEMMA 4.2. *Let $m>1$ be given. Then the function $\bar{p}(\varepsilon_1)$ in Lemma* 3.6 *can be taken as the right-hand side of* (4.7), *so that*

$$\lim_{\varepsilon_1 \searrow 0} \bar{p}(\varepsilon_1) = 4m \frac{\int_0^{\pi/2} \sin u \cos^2 \frac{u}{2} \left(\tan \frac{u}{2}\right)^{2/m} du}{\int_0^{\pi} \sin u \left(\tan \frac{u}{2}\right)^{2/m} du}.$$

One observes that Theorem 1.1 follows from Theorem 2.1, Theorem 3.7, and the previous lemma.

5. APPENDIX

Finally, in this section we provide the proofs of Lemmas 1.3, 1.4, and 4.1.

Proof of Lemma 1.3. Let (u, v) be a nontrivial solution of (1.1)–(1.3) in $(0, \delta]$ for some $0<\delta \ll 1$. We want to show that (u, v) satisfies (1.7) for some constants c and d. Integrating (1.2) twice with integrating factors $1/r$ and r respectively and using the fact that $v(0)=0$, we obtain

$$v = br^2 + \frac{1}{2} r^2 \int_\delta^r \frac{1}{\rho} f_2 \, d\rho - \frac{1}{2} \int_0^r \rho f_2 \, d\rho, \qquad f_2 \equiv -2(1-v) \sin^2 u. \tag{5.1}$$

Similarly, integrating (1.1) twice with the integrating factors r^{1-m} and r^{2m-1} respectively and using the fact that $u(0)=0$, we get

$$u = ar^m + \frac{1}{2m} r^m \int_\delta^r \frac{f_1}{\rho^{m+1}} \, d\rho - \frac{1}{2m} r^{-m} \int_0^r \rho^{m-1} f_1 \, d\rho, \tag{5.2}$$

where

$$f_1 = m^2[(1-v)^2 \sin u \cos u - u] - pr^2 \sin u \cos u.$$

We observe that

$$f_1 \leqslant K(|u| + |v|)\, |u| \qquad \text{for} \quad r \in [0, \delta], \tag{5.3}$$

where K is independent of $\delta \in (0, 1]$.

We now show that $u = O(r^m)$ as $r \searrow 0$. Define $w(\tilde{r}) = \sup_{0 \leqslant r \leqslant \tilde{r}} |u(r)|$. Taking the supremum of both sides of (5.2) from 0 to $\tilde{r}$ and using (5.3) yields

$$\begin{aligned} w(\tilde{r}) &\leqslant |a|\, \tilde{r}^m + \frac{1}{2m} \sup_{0 \leqslant r \leqslant \tilde{r}} \left\{ r^m \left(\int_r^{\tilde{r}} + \int_{\tilde{r}}^{\delta} \right) \frac{|f_1|}{\rho^{m+1}}\, d\rho \right\} + \frac{1}{2m^2} \sup_{0 \leqslant r \leqslant \tilde{r}} |f_1| \\ &\leqslant |a|\, \tilde{r}^m + \frac{1}{2m^2} \sup_{0 \leqslant r \leqslant \tilde{r}} |f_1| + \frac{1}{2m} \int_{\tilde{r}}^{\delta} \frac{|f_1|}{\rho^{m+1}}\, d\rho + \frac{1}{2m^2} \sup_{0 \leqslant r \leqslant \tilde{r}} |f_1| \\ &\leqslant |a|\, \tilde{r}^m + \frac{K}{m^2} \left[w(\tilde{r}) + \max_{0 \leqslant r \leqslant \tilde{r}} |v(r)| \right] w(\tilde{r}) + \frac{K}{2m}\, \tilde{r}^m \int_{\tilde{r}}^{\delta} \frac{[w + |v|]\, w}{\rho^{m+1}}\, d\rho. \end{aligned}$$

Since K is independent of δ, $w(0) = 0$, and $v(0) = 0$, we can set δ small enough such that $(K/m^2)[w(\tilde{r}) + \max_{0 \leqslant r \leqslant \tilde{r}} |v(r)|] < \frac{1}{2}$ for all $\tilde{r} \in (0, \delta]$. It then follows that

$$w(\tilde{r}) \leqslant 2\, |a|\, \tilde{r}^m + \frac{K}{m}\, \tilde{r}^m \int_{\tilde{r}}^{\delta} \frac{[w + |v|]\, w}{\rho^{m+1}}\, d\rho.$$

Dividing both sides by $\tilde{r}^m$, then taking the supremum of both sides from r to δ and defining $W(r) = \sup_{r \leqslant \tilde{r} \leqslant \delta} w(\tilde{r})/\tilde{r}^m$, we then obtain

$$W(r) \leqslant 2\, |a| + \frac{K}{m^2} \left[\delta^m W(r) + \sup_{r \leqslant \tilde{r} \leqslant \delta} \int_{\tilde{r}}^{\delta} \frac{|v|}{\rho}\, d\rho \right] W(r) \qquad \text{for} \quad r \in (0, \delta]. \tag{5.4}$$

Since $(u(0), v(0)) = (0, 0)$, $|f_2| \leqslant 1$ for small δ, so that, from (5.1), $|v - br^2| \leqslant r^2 \ln 1/r$ for $r \in [0, \delta]$ if $\delta \ll 1$. Therefore

$$\begin{aligned} \int_{\tilde{r}}^{\delta} \frac{|v|}{\rho}\, d\rho &\leqslant \int_{\tilde{r}}^{\delta} \left[|b|\, \rho + \rho \ln \frac{1}{\rho} \right] d\rho \\ &\leqslant \frac{1}{2}\, |b|\, \delta^2 + \delta^2 \ln \frac{1}{\delta} \leqslant \frac{1}{2}\, |v(\delta)| + \frac{3}{2}\, \delta^2 \ln \frac{1}{\delta} < \frac{m^2}{2K} \qquad \text{if} \quad \delta \ll 1. \end{aligned}$$

Hence from (5.4), we obtain

$$W(r) \leqslant 4\, |a| + \frac{2K}{m^2}\, \delta^m W^2(r) \qquad \text{for} \quad r \in (0, \delta]. \tag{5.5}$$

Note that $W(\delta) = w(\delta)/\delta^m = o(1)/\delta^m < 1/\delta^m$ if $\delta \ll 1$. Therefore, we can define $\bar{r} = \inf\{\hat{r} \in (0, \delta] : W(r) < 1/\delta^m \text{ in } (\hat{r}, \delta]\}$. We show that $\bar{r} = 0$. Suppose $\bar{r} > 0$. Then $W(\bar{r}) = 1/\delta^m$. Evaluating (5.5) at $\bar{r}$ and then multiplying both sides by δ^m yields $1 \leqslant 2\, |a|\, \delta^m + (2K/m^2)\, \delta^m$, which is impossible if δ is sufficiently small since from (5.2), $a\delta^m = u(\delta) + (1/2m)\, r^{-m} \int_0^\delta \rho^{m-1} f_1 \, d\rho = o(1)$ as $\delta \to 0$. Therefore $W(r) < 1/\delta^m$ in $(0, \delta]$ and hence $u(r) = O(r^m)$ as $r \searrow 0$.

Now substituting $u = O(r^m)$ and $v = O(r^2\, |\ln r|)$ into (5.1) yields $v = O(r^2)$ as $r \to 0$. Therefore $r^{m-1} f_1$ and $r f_2$ are integrable in $(0, \delta)$, so that (5.1) and (5.2) can be written as

$$u = cr^m + \frac{1}{2m}\, r^m \int_0^r \frac{f_1}{\rho^{m+1}}\, d\rho - \frac{1}{2m}\, r^{-m} \int_0^r \rho^{m-1} f_1 \, d\rho, \tag{5.6}$$

$$v = dr^2 + \frac{1}{2}\, r^2 \int_0^r \frac{1}{\rho}\, f_2 \, d\rho - \frac{1}{2} \int_0^r \rho f_2 \, d\rho, \tag{5.7}$$

where $c = b + \frac{1}{2} \int_\delta^0 f_2/\rho$ and $d = a + \frac{1}{2m} \int_\delta^0 f_1/\rho^{m+1}$. It is also easy to show from $u = O(r^m)$ and $v = O(r^2)$ that the terms involving integral in (5.6) and (5.7) are $O(r^{m+2})$ and $O(r^{2m+2})$ respectively and hence (1.7) follows.

To show the existence part of Lemma 1.3, it suffices to show that for any given c and d, (5.6)–(5.7) has a unique solution in $(0, \delta]$ for some $\delta > 0$. Let X be the set of the continuous functions $(u, v) : [0, \delta] \to R^2$ equipped with the norm $\|(u, v)\| = \sup_{0<r\leqslant\delta} |u(r)|/r^m + \sup_{0<r\leqslant\delta} |v(r)|/r^2$. Then X is a Banach space. For any (u, v) in X, define $(\tilde{u}, \tilde{v}) = T(u, v)$ by the right-hand side of (5.6) and (5.7) respectively. Then it is an easy exercise to check that if δ is sufficiently positive small, then T maps the ball $B(\delta) := \{(u, v) \in X : \|(u, v)\| \leqslant |c| + |d| + 1\}$ into itself and is a contraction. Therefore T has a unique fixed point in $B(\delta)$, which is a solution to (5.6)–(5.7). In addition, since the non-linear term in the v equation grows linearly in v, v will not blow-up in finite r. Also, the nonlinear term in the u equation implies that u will not below-up if v does not blow up. Hence, the solution will not below-up in finite r and can be extended to the interval $(0, \infty)$. The continuous dependence of (u, v) in (c, d, m, p) follows easily from (5.6), (5.7), and the smoothness of f_1 and f_2 in (u, v, m, p). This completes the proof of the lemma. ∎

Remark 5.1. Replacing f_1 and f_2 in (5.6) and (5.7) by their Taylor series about $(u, v) = (0, 0)$ and substituting $u = cr^m + O(r^{m+2})$ and $v = dr^2 + O(r^{2m+2})$, we get the next higher order expansion of (u, v) near $(0, 0)$. Substituting this expansion into (5.6) and (5.7) again, we can get next higher order expansion of u and v. Continuing this process we can get arbitrarily higher order expansions of (u, v) near $(0, 0)$.

Proof of Lemma 1.4. Since $u' > 0$ and $u < \pi$, $u(\infty)$ exists and either $u(\infty) = \pi$, or $u(\infty) = \pi/2$. The latter is impossible since otherwise, the equation for u implies $(ru')' > 0$ in $(0, \infty)$, which further implies that $u > u(1) + u'(1) \ln r \to \infty$ as $r \to \infty$. Therefore $u(\infty) = \pi$.

Now we prove (1.9) and (1.10). Defining $z=\sqrt{r}(\pi-u)$ and using $\sin u\cos u=-(1/\sqrt{r})\,z[1+O(z^2/r)]$, we can write (1.1) as

$$z''-\left\{p+O(\frac{z^2}{r})+O\left(\frac{1}{r^2}\right)\right\}z=0 \qquad \text{for } r\gg 1. \tag{5.8}$$

Since $z(r)=o(\sqrt{r})$ and $z>0$, a standard Liouville-Green approximation (cf. Chapter 6 of [4]) then gives that, as $r\to\infty$, $z(r)=Ce^{-\sqrt{p}r}[1+O(1/r)]$ and $z'(r)=-C\sqrt{p}e^{-\sqrt{p}r}[1+O(1/r)]$ for some positive constant $C>0$. The asymptotics (1.9) and (1.10) then follow by the relation between u and z.

Next we show (1.11) and (1.12). Since $v'>0$ and $v<1$ in $(0,\infty)$, integrating the equation $(v'/r)'=(-2(1-v)\sin^2 u)/r$ from r to ∞ twice yields

$$\frac{v'(r)}{r}=2\int_r^\infty \frac{(1-v(s))\sin^2 u(s)}{s}\,ds, \tag{5.9}$$

$$v(r)=v_\infty-\int_r^\infty (s^2-r^2)\,\frac{(1-v(s))\sin^2 u(s)}{s}\,ds. \tag{5.10}$$

Since $v\in(0,1)$ and $\sin u(s)=Ce^{-\sqrt{p}s}[1/\sqrt{s}+O(1/s^{3/2})]$, we see from (5.10) that $v=v_\infty-O(e^{-2\sqrt{p}r})$. Replacing v in the integrals in (5.9) and (5.10) by this approximation we then obtain (1.11) and (1.12).

Finally we show $v_\infty<1$. In fact, if $v_\infty=1$, then (5.10) and the monotonicity of v implies that

$$\begin{aligned}
1-v(r)&=\int_r^\infty (s^2-r^2)\,\frac{(1-v(s))\sin^2 u(s)}{s}\,ds\\
&\leqslant (1-v(r))\int_r^\infty (s^2-r^2)\,\frac{\sin^2 u(s)}{s}\,ds\\
&\leqslant \frac{1}{2}\,(1-v(r)) \qquad \text{if} \quad r \text{ is large enough.}
\end{aligned}$$

However, this would imply $v\equiv 1$ for all r large enough, which is impossible. Thus $v_\infty<1$, thereby completing the proof of Lemma 1.4. ▮

Proof of Lemma 4.1. We first prove that if $\lambda\in\mathbf{C}(p,\varepsilon)$, then $\lambda\leqslant\varepsilon^2(\ln\varepsilon)^2$ for $\varepsilon\ll 1$. We claim that

$$v(t)\geqslant\begin{cases}\left(\lambda-\dfrac{2\varepsilon^2}{m}\right)\dfrac{t^2}{2} & \text{for} \quad t\in[0,\,t_1]\\[2ex] \left(\lambda-\dfrac{2\varepsilon^2}{m}\right)\dfrac{t^2}{2}-\varepsilon^2t^2\ln\dfrac{t}{t_1} & \text{for} \quad t>t_1.\end{cases} \tag{5.11}$$

In fact, from (3.9) of Lemma 3.1 we have $\dot{v}(t)/t \geqslant \lambda - (4\varepsilon^2/m) \sin^2 u/2 \geqslant \lambda - 2\varepsilon^2/m$ for $t \in [0, t_1]$. On the other hand when $t > t_1$,

$$\begin{aligned}\frac{\dot{v}}{t} &= \frac{\dot{v}(t_1)}{t_1} - 2\varepsilon^2 \int_{t_1}^{t} \frac{(1-v)\sin^2 u}{s}\, ds \\ &\geqslant \lambda - \frac{2\varepsilon^2}{m} - 2\varepsilon^2 \int_{t_1}^{t} \frac{1}{s}\, ds = \lambda - \frac{2\varepsilon^2}{m} - 2\varepsilon^2 \ln \frac{t}{t_1}.\end{aligned}$$

The estimate (5.11) then follows by integration.

Now evaluating (5.11) at $t = 2/\sqrt{\lambda}$ yields

$$v(2/\sqrt{\lambda}) \geqslant \begin{cases} 2\left[1 - \dfrac{2\varepsilon^2}{m\lambda}\right] & \text{if } \dfrac{2}{\sqrt{\lambda}} \leqslant t_1, \\ 2\left[1 - \dfrac{2\varepsilon^2}{m\lambda} + \dfrac{\varepsilon^2}{\lambda} \ln \dfrac{\lambda t_1^2}{4}\right] & \text{if } \dfrac{2}{\sqrt{\lambda}} > t_1. \end{cases} \tag{5.12}$$

Since $t_1^m e^{\varepsilon\sqrt{p t_1}} \geqslant 1$, we have $t_1 \geqslant 1/(1 + \varepsilon\sqrt{p/m})$. Thus when $\varepsilon \ll 1$ we must have $\lambda < \varepsilon^2 (\ln \varepsilon)^2$ since otherwise $v(2/\sqrt{\lambda}) > 1$ by (5.12), which contradicts the definition of $\mathbf{C}(p, \varepsilon)$. This proves the first inequality of (4.1).

Once we know $\lambda \leqslant \varepsilon^2 \ln^2 \varepsilon$, the estimates (4.2) and (4.3) then follows directly from (3.7)–(3.9).

Finally, we prove (4.4). Since $(d/dt)(\dot{v}/t) = -2\varepsilon^2 (1-v) \sin^2 u < 0$, we know that $\dot{v} \leqslant \lambda t$ and $v \leqslant \lambda t^2/2$ for all $t > 0$. As $\lambda \leqslant \varepsilon^2 \ln^2 \varepsilon$, we then know that, as $\varepsilon \to 0$, $v \to 0$ uniformly in any compact set of $[0, \infty)$. Consequently, $u \to 2 \arctan t^m$ uniformly in any compact set of $[0, \infty)$. It then follows that for any finite $t > 0$,

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \left\{ \frac{\dot{v}}{\varepsilon^2 t} - \frac{\lambda}{\varepsilon^2} \right\} &= -\lim_{\varepsilon \searrow 0} \int_0^t 2(1-v)\sin^2 u \, \frac{ds}{s} \\ &= -\int_0^t 2 \sin^2(\arctan s^m)\, \frac{ds}{s} = -\frac{4t^{2m}}{m(1+t^{2m})}.\end{aligned}$$

Since $\dot{v} > 0$ for all $t > 0$, we then obtain

$$\liminf_{\varepsilon \searrow 0} \inf_{\lambda \in \mathbf{C}(p, \varepsilon)} \frac{\lambda m}{4\varepsilon^2} \geqslant \frac{t^{2m}}{1 + t^{2m}}$$

for any $t > 0$. Sending $t \to \infty$ then yields (4.4). This completes the proof of Lemma 4.1. ▮

ACKNOWLEDGMENTS

The first author thanks professor Stuart P. Hastings for many helpful discussions. The second author is thankful for the financial support of the National Science Foundation Grant DMS-9622872.

REFERENCES

1. I. L. Bogolubsky and A. A. Bogolubskaya, $2D$ topological solitons in the gauged easy-axis Heisenberg anti-ferromagnet model, *Phys. Lett. B* **395** (1997), 269–274.
2. I. L. Bogolubsky and A. A. Bogolubskaya, JINR, E5-96-73, Dubna (1996), *Ann. Fond. Louis de Broglie*, to appear.
3. J. M. McLeod and J. Serrin, The existence of similar solutions for some boundary layer problems, *Arch. Rational Mech. Anal.* **31** (1968), 288–303.
4. F. W. J. Olver, "Asymptotics and Special Functions," Academic Press, New York, 1974.