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Existence of Travelling Wave Solutions in a Tissue Interaction Model for Skin Pattern Formation

S. Ai
Department of Mathematical Sciences, University of Alabama in Huntsville,
Huntsville, AL 35899, USA
ais@email.uah.edu

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Summary. We study a tissue interaction model on skin pattern formation proposed by Cruywagen and Murray [*J. Nonlin. Sci.*, 2 (1992), 217–240]. We prove rigorously that the model has travelling wave solutions for all sufficiently large wave speeds, which were found numerically by Cruywagen, Maini, and Murray [*J. Math. Biol.*, 33 (1994), 193–210]. Our results also confirm the asymptotic expansions obtained for those solutions by formal perturbation analysis in the Cruywagen et al. article cited above.

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1. Introduction

In this paper we study the following boundary value problem:

$$\beta\varepsilon^2\frac{d^4\theta}{dz^4} - \mu\varepsilon\frac{d^3\theta}{dz^3} - \varepsilon\frac{d^2\theta}{dz^2} + \rho\theta = \rho\frac{d^2}{dz^2}\left\{\frac{n}{1+\nu(1-\varepsilon\tau\rho_1\theta)}\right\}, \tag{1.1}$$

$$\varepsilon\frac{d^2n}{dz^2} - \frac{dn}{dz} + n(1-n) = \varepsilon\alpha\frac{d}{dz}\left\{n\frac{d}{dz}\left(\frac{1-\varepsilon\tau\rho_1\theta}{1+\gamma n}\right)\right\}, \tag{1.2}$$

$$\lim_{z\to-\infty}(\theta, n) = (0, 0), \qquad \lim_{z\to\infty}(\theta, n) = (0, 1), \tag{1.3}$$

where β, μ, τ, ν, ρ, α, γ, and ε are positive parameters, $\rho_1 = 1/\rho$ and ε is small. The problem (1.1)–(1.3) was derived in [3] in seeking travelling wave solutions for a mathematical model proposed by Cruywagen and Murray [2] to account for the tissue interaction that leads to feather germ patterning in chick skin. We note that in [3], the coefficient $\beta\varepsilon^2$ of $d^4\theta/dz^4$ in (1.1) was misprinted as $\beta\varepsilon^4$, and the numerator $1 - \varepsilon\tau\rho_1\theta$

inside of the parentheses in the right-hand side of (1.2) was misprinted as $1 - \varepsilon\tau\rho\theta$. According to [2], [3], [4], the model assumes that tissue interaction between the epithelial and the dermal skin layers is mediated by two signal chemicals that are secreted in each layer respectively. Those chemicals diffuse across the basal lamina, a thin sheet separating the dermis and the epidermis, thus transmitting information between the layers. The model consists of seven coupled nonlinear partial differential equations: four to describe the production, degradation, and diffusion of the chemicals within and between layers; two conservation equations for dermal and epidermal cell densities; and a force balance equation for modelling stress in the epithelium. The full system is too complicated to do any useful mathematical analysis, and subsequently a special case of the model was considered in [2], [3], [4] based on the biological fact that the changes in cell strain and cell densities during pattern formation in many embryological situations are small. This implies that the epithelial dilation is small. Under some further assumptions the full model is reduced to a system of two partial differential equations, which, after nondimensionalization, takes the following form in the one-dimensional spatial case:

$$\beta\frac{\partial^4\tilde{\Theta}}{\partial x^4} - \mu\frac{\partial^3\tilde{\Theta}}{\partial t\partial x^2} - \frac{\partial^2\tilde{\Theta}}{\partial x^2} + \rho\tilde{\Theta} = \frac{\partial^2}{\partial x^2}\left\{\frac{\tau\tilde{N}}{1+\nu(1-\tilde{\Theta})}\right\}, \tag{1.4}$$

$$\frac{\partial^2\tilde{N}}{\partial x^2} - \frac{\partial\tilde{N}}{\partial t} + \tilde{N}(1-\tilde{N}) = \alpha\frac{\partial}{\partial x}\left\{\tilde{N}\frac{\partial}{\partial x}\left(\frac{1-\tilde{\Theta}}{1+\gamma\tilde{N}}\right)\right\}, \tag{1.5}$$

where $\tilde{\Theta}$ stands for the epithelial dilation and $\tilde{N}$ stands for the dermal cell density. We refer the reader to [2], [3], [4] and the references therein for the detailed derivation of the model and its biological background.

Cruywagen, Maini, and Murray [3] were looking for travelling wave fronts $(\tilde{\Theta}(x,t), \tilde{N}(x,t)) := (\tilde{\theta}(\tilde{z}), \tilde{n}(\tilde{z}))$ with $\tilde{z} = x + ct$ and the wave speed $c > 0$ for the system (1.4)–(1.5), where $(\tilde{\theta}, \tilde{n})$ satisfies

$$\beta\frac{d^4\tilde{\theta}}{d\tilde{z}^4} - \mu c\frac{d^3\tilde{\theta}}{d\tilde{z}^3} - \frac{d^2\tilde{\theta}}{d\tilde{z}^2} + \rho\tilde{\theta} = \tau\frac{d^2}{d\tilde{z}^2}\left\{\frac{\tilde{n}}{1+\nu(1-\tilde{\theta})}\right\}, \tag{1.6}$$

$$\frac{d^2\tilde{n}}{d\tilde{z}^2} - c\frac{d\tilde{n}}{d\tilde{z}} + \tilde{n}(1-\tilde{n}) = \alpha\frac{d}{d\tilde{z}}\left\{\tilde{n}\frac{d}{d\tilde{z}}\left(\frac{1-\tilde{\theta}}{1+\gamma\tilde{n}}\right)\right\}, \tag{1.7}$$

$$\lim_{z\to-\infty}(\tilde{\theta}, \tilde{n}) = (0,0), \qquad \lim_{z\to-\infty}(\tilde{\theta}, \tilde{n}) = (0,1). \tag{1.8}$$

Based on numerical simulations, the local stability analysis at the equilibria of (1.6)–(1.7), and the observation that (1.7) decouples from (1.6) when $\alpha = 0$, which is the well-studied Fisher equation exhibiting wave front solutions for all wave speeds $c \geq 2$, they conjectured that (1.6)–(1.8) has solutions for sufficiently large c. Using the rescalings,

$$\tilde{z} = cz, \qquad \tilde{\theta}(\tilde{z}) = \frac{\tau}{\rho c^2}\theta(z), \qquad \tilde{n}(\tilde{z}) = n(z), \qquad \varepsilon = \frac{1}{c^2},$$

they reduced (1.6)–(1.8) to (1.1)–(1.3) with ε sufficiently small, and then applied regular series expansions of the form $\theta(z) = \theta_0(z) + \varepsilon\theta_1(z) + \cdots$ and $n(z) = n_0(z) + \varepsilon n_1(z) + \cdots$

to obtain an approximation to each wave front solution of (1.1)–(1.3). The $O(1)$ terms of the above expansions satisfy

$$\theta_0 = \frac{1}{1+\nu}\frac{d^2 n_0}{dz^2}, \qquad \frac{dn_0}{dz} = n_0(1-n_0). \tag{1.9}$$

By imposing the condition $n_0(0) = \frac{1}{2}$, they obtained $n_0(z) = e^z/(1+e^z)$, which clearly satisfies $n_0' > 0$, $\lim_{z\to-\infty} n_0(z) = 0$, and $\lim_{z\to\infty} n_0(z) = 1$. Since the above argument was based on purely formal perturbation analysis, it is the purpose of this paper to give a rigorous discussion of the existence and asymptotic behavior of such solutions. The main result of the paper is as follows:

Theorem 1.1. *Let $n_0(z) = e^z/(1+e^z)$ and θ_0 be defined in (1.9). If ε is sufficiently small, then there exists a solution $(\theta_\varepsilon, n_\varepsilon)$ to (1.1)–(1.3) that satisfies $n_\varepsilon > 0$ on $(-\infty, \infty)$ and the following:*

(i) *for any nonnegative integers j and $z \in (-\infty, \infty)$,*

$$\left|\frac{d^j}{dz^j}(\theta_\varepsilon(z) - \theta_0(z))\right| \le C_j\varepsilon, \qquad \left|\frac{d^j}{dz^j}(n_\varepsilon(z) - n_0(z))\right| \le C_j\varepsilon, \tag{1.10}$$

where $C_j > 0$ is a constant independent of ϵ;

(ii) *as $z \to -\infty$,*

$$\begin{pmatrix} \theta_\varepsilon(z) \\ \theta_\varepsilon'(z) \\ \theta_\varepsilon''(z) \\ \theta_\varepsilon'''(z) \\ n_\varepsilon(z) \\ n_\varepsilon'(z) \end{pmatrix} \sim c_{03}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \lambda_{03} \end{pmatrix} e^{\lambda_{03}z} + c_{04}\begin{pmatrix} 1 \\ \lambda_{04} \\ \lambda_{04}^2 \\ \lambda_{04}^3 \\ 0 \\ 0 \end{pmatrix} e^{\lambda_{04}z} + c_{05}\begin{pmatrix} 1 \\ \lambda_{05} \\ \lambda_{05}^2 \\ \lambda_{05}^3 \\ 0 \\ 0 \end{pmatrix} e^{\lambda_{05}z}, \tag{1.11}$$

where c_{0j} and λ_{0j} $(j = 3, 4, 5)$ are real numbers, $c_{03} > 0$, and

$$\lambda_{03} \sim 1, \qquad \lambda_{04} \sim \sqrt[3]{\frac{\rho}{\mu\varepsilon}}, \qquad \lambda_{05} \sim \frac{\mu}{\beta\varepsilon} \qquad \text{as } \varepsilon \to 0, \tag{1.12}$$

while, as $z \to \infty$,

$$\begin{pmatrix} \theta_\varepsilon(z) \\ \theta_\varepsilon'(z) \\ \theta_\varepsilon''(z) \\ \theta_\varepsilon'''(z) \\ n_\varepsilon(z) - 1 \\ n_\varepsilon'(z) \end{pmatrix} \sim c_{11}\Re\left\{\begin{pmatrix} d_2 \\ d_2\lambda_{12} \\ d_2\lambda_{12}^2 \\ d_2\lambda_{12}^3 \\ 1 \\ \lambda_{12} \end{pmatrix} e^{ib_{11}z}\right\} e^{a_{11}z} + c_{12}\Im\left\{\begin{pmatrix} d_2 \\ d_2\lambda_{12} \\ d_2\lambda_{12}^2 \\ d_2\lambda_{12}^3 \\ 1 \\ \lambda_{12} \end{pmatrix} e^{ib_{11}z}\right\} e^{a_{11}z}$$

$$+ c_{13}\begin{pmatrix} d_3 \\ d_3\lambda_{13} \\ d_3\lambda_{13}^2 \\ d_3\lambda_{13}^3 \\ 1 \\ \lambda_{13} \end{pmatrix} e^{\lambda_{13}z}, \tag{1.13}$$

where $d_j = \frac{\lambda_{1j}^2 - L\lambda_{1j}/\varepsilon - L/\varepsilon}{\lambda_{1j}^2} \neq 0$ *(*$j = 2, 3$*),* $L = \frac{(1+\gamma)^2}{(1+\gamma)^2 + \alpha\gamma}$*,* $\lambda_{12} = a_{11} + ib_{11}$*,* c_{1j} *(*$j = 1, 2, 3$*),* a_{11}*,* b_{11}*, and* λ_{13} *are real numbers such that* $|c_{11}| + |c_{12}| + |c_{13}| \neq 0$*, and*

$$a_{11} \sim -\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}, \qquad b_{11} \sim \frac{\sqrt{3}}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}, \qquad \lambda_{13} \sim -1 \qquad \text{as } \varepsilon \to 0. \tag{1.14}$$

Remark 1.1. (a) Transforming $\theta_\varepsilon, n_\varepsilon, z$ back to the variables $\tilde{\theta}_\varepsilon, \tilde{n}_\varepsilon, \tilde{z}$, (1.10) yields that for any given nonnegative integer j, if c is sufficiently large, then for $\tilde{z} \in (-\infty, \infty)$,

$$\left|\frac{d^j\tilde{\theta}_\varepsilon}{d\tilde{z}^j} - \frac{\tau}{\rho(1+\nu)}\frac{d^{j+2}}{d\tilde{z}^{j+2}}\left\{\frac{e^{\frac{1}{c}\tilde{z}}}{1+e^{\frac{1}{c}\tilde{z}}}\right\}\right| \le \frac{C_j}{c^{j+4}}, \quad \left|\frac{d^j\tilde{n}_\varepsilon}{d\tilde{z}^j} - \frac{d^j}{d\tilde{z}^j}\left\{\frac{e^{\frac{1}{c}\tilde{z}}}{1+e^{\frac{1}{c}\tilde{z}}}\right\}\right| \le \frac{C_j}{c^{j+2}}.$$

(b) Biologically, the dermal cell density n satisfies $0 < n_\varepsilon < 1$. The second estimate in (1.10) implies that for any fixed $z \in (-\infty, \infty)$, $n_\varepsilon(z) < 1$ if ε is sufficiently small. It follows from (1.11) that $n_\varepsilon'/n_\varepsilon \to \lambda_{03}$ as $z \to -\infty$, which resembles the travelling wave fronts for the Fisher equation (see [5]).

(c) It is also expected biologically the dilation θ_ε does not approach the zero steady state in an oscillatory manner as $z \to \pm\infty$. This is confirmed by (1.11) as $z \to -\infty$. If one shows that $n_\varepsilon < 1$ near $z = \infty$, then it follows that $c_{13} < 0$ in (1.13) and hence θ_ε does not oscillate as $z \to \infty$.

We note that by formally setting $v(z) = \int_{-\infty}^z \int_{-\infty}^\xi \theta(\eta)\, d\eta d\xi$ and integrating (1.1) over $(-\infty, z)$ two times we transform (1.1)–(1.3) into the following problem:

$$\beta\varepsilon^2\frac{d^4v}{dz^4} - \mu\varepsilon\frac{d^3v}{dz^3} - \varepsilon\frac{d^2v}{dz^2} + \rho v = \frac{\rho n}{1 + \nu(1 - \varepsilon\tau\rho_1 v'')}, \tag{1.15}$$

$$\varepsilon\frac{d^2n}{dz^2} - \frac{dn}{dz} + n(1-n) = \varepsilon\alpha\frac{d}{dz}\left\{n\frac{d}{dz}\left(\frac{1 - \varepsilon\tau\rho_1 v''}{1+\gamma n}\right)\right\}, \tag{1.16}$$

$$\lim_{z\to-\infty}(v, n) = (0, 0), \qquad \lim_{z\to\infty}(v, n) = \left(\frac{1}{1+\nu}, 1\right), \tag{1.17}$$

where $v'' := d^2v/dz^2$. Note that this transformation does not change the left-hand side of the equation (1.1) while its right-hand side becomes simpler. It is easy to check that if (v, n) is a solution of (1.15)–(1.17), then $(\theta, n) := (v'', n)$ gives a solution of (1.1)–(1.3). Therefore, the existence of $(\theta_\varepsilon, n_\varepsilon)$ satisfying (i) in Theorem 1.1 follows from the following theorem:

Theorem 1.2. *If* ε *is sufficiently small, then there exists a solution* $(v_\varepsilon, n_\varepsilon)$ *to (1.15)–(1.17) that satisfies, for any nonnegative integers* j *and* $z \in (-\infty, \infty)$*,*

$$\left|\frac{d^j}{dz^j}\left(v_\varepsilon(z) - \frac{n_0(z)}{1+\nu}\right)\right| \le C_j\varepsilon, \qquad \left|\frac{d^j}{dz^j}\Big(n_\varepsilon(z) - n_0(z)\Big)\right| \le C_j\varepsilon, \tag{1.18}$$

where $C_j > 0$ *is a constant independent of* ε*.*

We prove Theorem 1.2 in Section 2 by the contraction mapping theorem and Theorem 1.1 in Section 3. The asymptotic behavior in (1.11) and (1.13) follows from an application of the stable manifold theorem to an equivalent first-order system of (1.1)–(1.2). We show that if ε is sufficiently small, then $w_0 := (0,0,0,0,0,0)$ and $w_1 := (0,0,0,0,1,0)$ are the only equilibria of this system, and the unstable manifold at w_0 is four-dimensional and the stable manifold at w_1 is three-dimensional. λ_{0j} $(j = 3,4,5)$ and the vectors in the right-hand side of (1.11) are the eigenvalues with the positive real parts at w_0 and their corresponding eigenvectors. $\lambda_{11} = a_{11} - ib_{11}$, $\lambda_{12} = a_{11} + ib_{11}$, and λ_{13} in (1.13) are the eigenvalues with negative real parts at w_1, and the vectors in the right-hand side of (1.13) are the eigenvectors associated with λ_{12} and λ_{13}. (1.12) and (1.14) provide the estimates for those eigenvalues as $\varepsilon \to 0$. The existence of those eigenvalues and their asymptotic formulas are presented in two lemmas in the Appendix. In order to show $n_\varepsilon > 0$ on $(-\infty, \infty)$, we use an argument similar to the phase-plane argument used for wave front solutions of the Fisher equation (see [5], [8] for discussions on the Fisher equation).

In the rest of the paper, the dependence on ε for solutions of (1.1)–(1.2) or (1.15)–(1.16) is suppressed.

2. Proof of Theorem 1.2

For convenience, we let $BC(-\infty,\infty)$ be the Banach space of all continuous and bounded functions on $(-\infty,\infty)$ with the norm $|f|_0 = \sup\{|f(z)| : z \in (-\infty,\infty)\}$ for $f \in BC(-\infty,\infty)$. For given positive numbers N, σ, and ω, we define

$$Y_N = \{n \in BC(-\infty,\infty) : |n|_0 \le N, \quad \lim_{z\to-\infty} n(z) = 0, \quad \lim_{z\to\infty} n(z) = 1\},$$

$$Z_\sigma = \{n_1 \in BC(-\infty,\infty) : |n_1|_0 \le \sigma, \quad \lim_{z\to\pm\infty} n_1(z) = 0\},$$

$$W_\omega = \left\{ v \in BC(-\infty,\infty) : \lim_{z\to-\infty} v(z) = 0, \quad \lim_{z\to\infty} v(z) = \frac{\omega}{1+\nu} \right\}.$$

Then, it is easy to verify that Y_N, Z_σ, and W_ω are closed sets in $BC(-\infty,\infty)$.

We prove Theorem 1.2 by three steps. First, for any given positive number N and $n \in Y_N$, we show that for sufficiently small ε, the equation (1.15) has a unique solution $v := \mathcal{V}(n) \in W_1$, and $\mathcal{V}(n)$ as a mapping from Y_N to W_1 is Lipschitz continuous with respect to n. We complete this step by Lemmas 5.2, 2.1, and 2.2 and Corollary 2.1.

Secondly, we write the equation (1.16) as an equivalent system (2.28) for (m,n). Since we expect that n' is bounded, by formally sending $\varepsilon \to 0$ in the second equation of (2.28), we see $m - n \to 0$. We also expect $n \to n_0$ as $\varepsilon \to 0$. Hence if we set $m = n_0 + m_1$ and $n = m + n_1$, we expect that both m_1 and n_1 which satisfy the equations (2.29) and (2.30) are small. Note that the equation (2.29) does not depend on ε, v and its derivatives. We show in Lemma 2.3 that for sufficiently small $\sigma > 0$ there is a number $\sigma_1 = O(\sigma)$ such that for any function $n_1 \in Z_\sigma$ there is a unique $m_1 := \mathcal{M}_1(n_1) \in Z_{\sigma_1}$ satisfying (2.29). We further show that $\mathcal{M}_1$ as a mapping from Z_σ to Z_{σ_1} is Lipschitz continuous with respect to n_1.

Finally we show in Lemma 2.4 that there exists a small $\sigma > 0$ such that for sufficiently small ε, there is a unique $n_1 \in Z_\sigma$ that satisfies the equation (2.30) with m_1, n, v, and the derivatives of v in the right-hand side of (2.30) replaced by $\mathcal{M}_1(n_1), \mathcal{N}(n_1) := n_0 + \mathcal{M}_1(n_1)+n_1, \mathcal{V}(\mathcal{N}(n_1))$ and its derivatives. It follows that $(v, n) := (\mathcal{V}(\mathcal{N}(n_1)), \mathcal{N}(n_1))$ is a solution to (1.15)–(1.17).

We start by transforming (1.15) into equivalent integral equations. To do so, we consider the nonhomogeneous equation

$$\beta\varepsilon^2 \frac{d^4 v}{dz^4} - \mu\varepsilon \frac{d^3 v}{dz^3} - \varepsilon \frac{d^2 v}{dz^2} + \rho v = f(z), \tag{2.1}$$

where $f \in BC(-\infty, \infty)$. Write an equivalent system to (2.1) as $\phi' = A\phi + F(z)$, where $\phi := (v, v', v'', v''')^t$, A is the corresponding 4×4 constant coefficient matrix, and $F(z) = (0, 0, 0, f(z)/\beta\varepsilon^2)^t$. The characteristic equation for A is $p(\lambda) = \beta\varepsilon^2\lambda^4 - \mu\varepsilon\lambda^3 - \varepsilon\lambda^2 + \rho = 0$. It follows from Lemma 5.2 that if ε is sufficiently small, then $p(\lambda) = 0$ has two complex roots $\lambda_1 = a - ib$ and $\lambda_2 = a + ib$ and two real roots $0 < \lambda_3 < \lambda_4$. Note that the eigenvectors of A associated with the eigenvalues λ_j $(i = 1, 2, 3, 4)$ are $(1, \lambda_i, \lambda_i^2, \lambda_i^3)^t$. We have

$$T^{-1}AT = \Lambda := \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

$$\text{where } T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ a & b & \lambda_3 & \lambda_4 \\ a^2 - b^2 & 2ab & \lambda_3^2 & \lambda_4^2 \\ a^3 - 3ab^2 & 3a^2b - b^3 & \lambda_3^3 & \lambda_4^3 \end{pmatrix},$$

where the first two columns of T are the real and the imaginary parts of the complex eigenvector $(1, \lambda_2, \lambda_2^2, \lambda_2^3)^t$, respectively. Let $\phi = Tx$ with $x = (x_1, x_2, x_3, x_4)^t$. It follows that

$$x_1' = ax_1 + bx_2 + \frac{\alpha_1}{\beta\varepsilon^2} f(z), \qquad x_2' = -bx_1 + ax_2 + \frac{\alpha_2}{\beta\varepsilon^2} f(z), \tag{2.2}$$

$$x_3' = \lambda_3 x_3 + \frac{\alpha_3}{\beta\varepsilon^2} f(z), \qquad x_4' = \lambda_4 x_4 + \frac{\alpha_4}{\beta\varepsilon^2} f(z), \tag{2.3}$$

where $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^t$ is the last column of T^{-1} given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} := T^{-1}e_4 = \begin{pmatrix} \frac{\lambda_4+\lambda_3-2a}{[(\lambda_4-a)^2+b^2][(\lambda_3-a)^2+b^2]} \\ \frac{(\lambda_3-a)\lambda_4+a^2-b^2-a\lambda_3}{b[(\lambda_4-a)^2+b^2][(\lambda_3-a)^2+b^2]} \\ \frac{1}{(\lambda_3-\lambda_4)[(\lambda_3-a)^2+b^2]} \\ \frac{1}{(\lambda_4-\lambda_3)[(\lambda_4-a)^2+b^2]} \end{pmatrix}, \quad \text{where} \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.4}$$

Then an easy exercise shows that the unique bounded solution of (2.2)–(2.3) over $(-\infty, \infty)$ is given by

$$x_1(z) = \frac{1}{\beta\varepsilon^2} \int_{-\infty}^{z} e^{a(z-s)}[\alpha_1 \cos b(z-s) + \alpha_2 \sin b(z-s)] f(s)\, ds, \tag{2.5}$$

$$x_2(z) \;=\; \frac{1}{\beta\varepsilon^2}\int_{-\infty}^{z} e^{a(z-s)}[-\alpha_1\sin b(z-s)+\alpha_2\cos b(z-s)]f(s)\,ds, \tag{2.6}$$

$$x_3(z) \;=\; -\frac{\alpha_3}{\beta\varepsilon^2}\int_{z}^{\infty} e^{\lambda_3(z-s)}f(s)\,ds, \tag{2.7}$$

$$x_4(z) \;=\; -\frac{\alpha_4}{\beta\varepsilon^2}\int_{z}^{\infty} e^{\lambda_4(z-s)}f(s)\,ds, \tag{2.8}$$

and furthermore, if $f \in W_\rho$, then

$$x(z) \to \begin{cases} 0, & \text{as } z \to -\infty, \\ -\frac{\rho}{\beta\varepsilon^2(1+\nu)}\Lambda^{-1}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^t, & \text{as } z \to \infty. \end{cases} \tag{2.9}$$

From (5.5) and (2.4) it follows that, as $\varepsilon \to 0$,

$$\alpha_1 \sim \frac{1}{\lambda_4[(\lambda_3-a)^2+b^2]}, \qquad \alpha_2 \sim \frac{\lambda_3-a}{b\lambda_4[(\lambda_3-a)^2+b^2]},$$
$$\alpha_3 \sim \frac{-1}{\lambda_4[(\lambda_3-a)^2+b^2]}, \qquad \alpha_4 \sim \frac{1}{\lambda_4^3}, \tag{2.10}$$

and hence there is a constant $C > 0$ independent of ε such that if ε is sufficiently small, then

$$|\alpha_1|+|\alpha_2|+|\alpha_3| \le C\varepsilon^{\frac{5}{3}}, \qquad |\alpha_4| \le C\varepsilon^3,$$

which together with (2.5)–(2.8) yields

$$|x_1|_0+|x_2|_0+|x_3|_0 \le C|f|_0, \qquad |x_4|_0 \le C\varepsilon^2|f|_0, \tag{2.11}$$

where the constant C might be changed but is still independent of ε. Substituting back to the original variable v, we have the following result:

Lemma 2.1. *If ε is sufficiently small, then for any $f \in BC(-\infty,\infty)$, the equation (2.1) has a unique bounded solution $v(z)$ defined for $z \in (-\infty,\infty)$ by*

$$v(z) \;=\; x_1(z)+x_3(z)+x_4(z), \tag{2.12}$$

$$v'(z) \;=\; ax_1(z)+bx_2(z)+\lambda_3x_3(z)+\lambda_4x_4(z), \tag{2.13}$$

$$v''(z) \;=\; (a^2-b^2)x_1(z)+2abx_2(z)+\lambda_3^2x_3(z)+\lambda_4^2x_4(z), \tag{2.14}$$

$$v'''(z) \;=\; (a^3-3ab^2)x_1(z)+(3a^2b-b^3)x_2(z)+\lambda_3^3x_3(z)+\lambda_4^3x_4(z), \tag{2.15}$$

where α_i and x_i, $i = 1, 2, 3, 4$, are given in (2.4)–(2.8), such that

$$|v|_0+\varepsilon^{\frac{1}{3}}|v'|_0+\varepsilon^{\frac{2}{3}}|v''|_0+\varepsilon|v'''|_0 \le C|f|_0, \tag{2.16}$$

where $C > 0$ is a constant independent of ε and f. Furthermore, if $f(z) \in W_\rho$, then

$$\phi(z) := (v,v',v'',v''')^t(z) \to \begin{cases} 0, & \text{as } z \to -\infty, \\ (\frac{1}{1+\nu},0,0,0)^t, & \text{as } z \to \infty. \end{cases} \tag{2.17}$$

We note that (2.17) follows from (2.9). To see this, we only need to check the limit of $\phi(z)$ as $z \to \infty$, which results from noting that, as $z \to \infty$,

$$\begin{aligned}\phi(z) &= Tx(z) \to -\frac{\rho}{\beta\varepsilon^2(1+\nu)} T\Lambda^{-1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^t \\ &= -\frac{\rho}{\beta\varepsilon^2(1+\nu)}(A^{-1}T)(T^{-1}e_4) \\ &= -\frac{\rho}{\beta\varepsilon^2(1+\nu)}A^{-1}e_4 = -\frac{\rho}{\beta\varepsilon^2(1+\nu)}\left(-\frac{\beta\varepsilon^2}{\rho}, 0, 0, 0\right)^t \\ &= \left(\frac{1}{1+\nu}, 0, 0, 0\right)^t. \end{aligned} \tag{2.18}$$

Let X_ε be the Banach space of continuous and bounded vector-valued functions $x = (x_1, x_2, x_3, x_4)^t \in [C(-\infty, \infty)]^4$ with the norm $|x|_\varepsilon := |x_1|_0 + |x_2|_0 + |x_3|_0 + \varepsilon^{-2}|x_4|_0$. We define four linear mappings $V(x)$, $V_1(x)$, $V_2(x)$, and $V_3(x)$ from X_ε to $BC(-\infty, \infty)$ by the right-hand sides of (2.12)–(2.15) respectively. It follows from (2.10) and (5.5) that, for $x \in X_\varepsilon$,

$$\begin{aligned}|V(x)|_0 &\le |x_1|_0 + |x_3|_0 + |x_4|_0 \le |x|_\varepsilon, \\ |V_1(x)|_0 &\le C(\varepsilon^{-\frac{1}{3}}|x_1|_0 + \varepsilon^{-\frac{1}{3}}|x_2|_0 + \varepsilon^{-\frac{1}{3}}|x_3|_0 + \varepsilon^{-1}|x_4|_0) \le C\varepsilon^{-\frac{1}{3}}|x|_\varepsilon, \\ |V_2(x)|_0 &\le C(\varepsilon^{-\frac{2}{3}}|x_1|_0 + \varepsilon^{-\frac{2}{3}}|x_2|_0 + \varepsilon^{-\frac{2}{3}}|x_3|_0 + \varepsilon^{-2}|x_4|_0) \le C\varepsilon^{-\frac{2}{3}}|x|_\varepsilon, \\ |V_3(x)|_0 &\le C(\varepsilon^{-1}|x_1|_0 + \varepsilon^{-1}|x_2|_0 + \varepsilon^{-1}|x_3|_0 + \varepsilon^{-3}|x_4|_0) \le C\varepsilon^{-1}|x|_\varepsilon,\end{aligned}$$

and so

$$|V(x)|_0 + \varepsilon^{\frac{1}{3}}|V_1(x)|_0 + \varepsilon^{\frac{2}{3}}|V_2(x)|_0 + \varepsilon|V_3(x)|_0 \le C|x|_\varepsilon. \tag{2.19}$$

Similarly we have, for x and $\bar{x}$ in X_ε,

$$\begin{aligned}&|V(x) - V(\bar{x})|_0 + \varepsilon^{\frac{1}{3}}|V_1(x) - V_1(\bar{x})|_0 + \varepsilon^{\frac{2}{3}}|V_2(x) - V_2(\bar{x})|_0 \\ &\quad + \varepsilon|V_3(x) - V_3(\bar{x})|_0 \le C|x - \bar{x}|_\varepsilon.\end{aligned} \tag{2.20}$$

Moreover, for any $x \in X_\varepsilon$, the same argument used in (2.18) yields that $(V_1(x), V_2(x), V_3(x), V_4(x))^t(z)$ has the same limits as $z \to \pm\infty$ as those of $\phi(z)$ given in (2.17), and, in particular, $V_2(x)(z) \to 0$ as $z \to \pm\infty$. Now we are ready to show the next lemma:

Lemma 2.2. *For any positive number N, there exist an $\varepsilon_0 = \varepsilon_0(N) > 0$ and a constant $C^* > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $n \in Y_N$ there exists a unique $x := \mathcal{X}(n) \in B_\varepsilon(C^*N) := \{x \in X_\varepsilon : |x|_\varepsilon \le C^*N,$ and x satisfies (2.9)$\}$ satisfying (2.2)–(2.3), where f in (2.2) and (2.3) is defined by*

$$f(z) := \tilde{f}(n(z), V_2(x)(z)), \qquad \tilde{f}(s_1, s_2) := \frac{\rho s_1}{1 + \nu(1 - \varepsilon\tau\rho_1 s_2)}. \tag{2.21}$$

Moreover,

$$|\mathcal{X}(n)|_\varepsilon \le C^*|n|_0, \qquad |\mathcal{X}(n) - \mathcal{X}(\bar{n})|_\varepsilon \le C^*|n - \bar{n}|_0, \tag{2.22}$$

for any n and $\bar{n}$ in Y_N.

Proof. We use the contraction mapping theorem to prove the lemma. First we define the mapping $\mathcal{P}$ on $B_\varepsilon(C^*N)$ by $\mathcal{P}(x) = (\mathcal{P}_1(x), \mathcal{P}_2(x), \mathcal{P}_3(x), \mathcal{P}_4(x))$, where $\mathcal{P}_i(x)$, $i = 1, 2, 3, 4$ are defined by the right-hand sides of (2.5), (2.6), (2.7), and (2.8) respectively, with f inside of each integral in (2.5), (2.6), (2.7), and (2.8) being defined by (2.21). The constant C^* will be chosen shortly. We note that $B_\varepsilon(C^*N)$ is a closed set of X_ε. It follows from (2.19) and (2.21) that for any $x \in X_\varepsilon$,

$$|f|_0 = \max_{-\infty<z<\infty} |\tilde{f}(n(z), V_2(x)(z))| \le \frac{\rho|n|_0}{1 + \nu(1 - C\varepsilon^{\frac{1}{3}}|x|_\varepsilon)}.$$

The constant C may have been changed from that in (2.19). It is easy to see that $f(z) \to 0$ as $z \to -\infty$ and $f(z) \to \rho/(1+\nu)$ as $z \to \infty$, namely, $f \in W_\rho$. Hence, $\mathcal{P}(x)$ is in X_ε and satisfies (2.9). Moreover, it follows from (2.11) that if $|x|_\varepsilon \le \frac{\rho CN}{1+\nu/2}$ and $\varepsilon < (\frac{1+\nu/2}{2\rho C^2 N})^3$, then

$$|\mathcal{P}(x)|_\varepsilon \le C|f|_0 \le \frac{\rho C|n|_0}{1 + \nu(1 - C\varepsilon^{\frac{1}{3}}|x|_\varepsilon)} \le \frac{\rho C|n|_0}{1 + \nu/2} \le \frac{\rho CN}{1 + \nu/2}. \tag{2.23}$$

Let $C^* = \frac{\rho C}{1+\nu/2}$. We see that $\mathcal{P}(x) \in B_\varepsilon(C^*N)$ if $x \in B_\varepsilon(C^*N)$. This shows that if ε is sufficiently small, then $\mathcal{P}$ maps $B_\varepsilon(C^*N)$ into itself.

Next we show that $\mathcal{P}$ is a contraction mapping on $B_\varepsilon(C^*N)$. We note that for $x \in B_\varepsilon(C^*N)$, using $\frac{\nu}{1+\nu/2} \le 2$ and $\rho\rho_1 = 1$, we have

$$\left|\frac{\partial \tilde{f}}{\partial s_2}\right|_0 = \left|\frac{\varepsilon\nu\tau n}{[1 + \nu(1 - \varepsilon\tau\rho_1 V_2(x))]^2}\right|_0 \le \frac{\varepsilon\nu\tau N}{[1 + \nu(1 - C\varepsilon^{\frac{1}{3}}|x|_\varepsilon)]^2} \le 2\tau\varepsilon N. \tag{2.24}$$

Hence, by the mean value theorem and (2.20) we have, for any x and $\bar{x}$ in $B_\varepsilon(C^*N)$,

$$|\tilde{f}(n, V_2(x)) - \tilde{f}(n, V_2(\bar{x}))|_0 \le 2\varepsilon\tau N|V_2(x) - V_2(\bar{x})|_0 \le C\varepsilon^{\frac{1}{3}}N|x - \bar{x}|_\varepsilon.$$

Therefore by (2.11) we obtain

$$\sum_{i=1}^{3} |\mathcal{P}_i(x) - \mathcal{P}_i(\bar{x})|_0 \ \le\ C|\tilde{f}(n, V_2(x)) - \tilde{f}(n, V_2(\bar{x}))|_0 \le C^2\varepsilon^{\frac{1}{3}}N|x - \bar{x}|_\varepsilon,$$

$$|\mathcal{P}_4(x) - \mathcal{P}_4(\bar{x})|_0 \ \le\ C\varepsilon^2|\tilde{f}(n, V_2(x)) - \tilde{f}(n, V_2(\bar{x}))|_0 \le C^2N\varepsilon^{\frac{1}{3}}\varepsilon^2|x - \bar{x}|_\varepsilon,$$

and so $|\mathcal{P}(x) - \mathcal{P}(\bar{x})|_\varepsilon \le C^2N\varepsilon^{\frac{1}{3}}|x - \bar{x}|_\varepsilon$. Therefore, $\mathcal{P}$ is a contraction over $B_\varepsilon(C^*N)$ if ε is sufficiently small, and so $\mathcal{P}$ has a unique fixed point $x := \mathcal{X}(n) \in B_\varepsilon(C^*N)$, which gives a solution to (2.2)–(2.3). Furthermore, it follows from (2.23) that $|\mathcal{X}(n)|_\varepsilon \le C^*|n|_0$.

To show the second inequality in (2.22), we note that, for any $n \in Y_N$, $x \in B_\varepsilon(C^*N)$,

$$\left|\frac{\partial \tilde{f}}{\partial s_1}\right|_0 = \left|\frac{\rho}{1 + \nu(1 - \varepsilon\tau\rho_1 V_2(x))}\right|_0 \le \left|\frac{\rho}{1 + \nu(1 - C\varepsilon^{\frac{1}{3}}|x|_\varepsilon)}\right| \le \frac{\rho}{1 + \nu/2}.$$

Then using the mean value theorem, (2.20) and (2.24), we get, for any n and $\bar{n}$ in Y_N,

$$\begin{aligned}|\tilde{f}(n, \mathcal{X}(n)) - \tilde{f}(\bar{n}, \mathcal{X}(\bar{n}))|_0 \ &\le\ 2\varepsilon\tau N|V_2(\mathcal{X}(n)) - V_2(\mathcal{X}(\bar{n}))|_0 + \frac{\rho}{1 + \nu/2}|n - \bar{n}|_0 \\ &\le\ 2\varepsilon^{\frac{1}{3}}\tau NC|\mathcal{X}(n) - \mathcal{X}(\bar{n})|_\varepsilon + \frac{\rho}{1 + \nu/2}|n - \bar{n}|_0,\end{aligned}$$

and then by using (2.11) we have

$$
\begin{aligned}
|\mathcal{X}(n)-\mathcal{X}(\bar{n})|_\varepsilon &= \sum_{i=1}^{3}|\mathcal{X}_i(n)-\mathcal{X}_i(\bar{n})|_0+\varepsilon^{-2}|\mathcal{X}_4(n)-\mathcal{X}_4(\bar{n})|_0\\
&\le 2C|\tilde{f}(n,\mathcal{X}(n))-\tilde{f}(\bar{n},\mathcal{X}(\bar{n}))|_0\\
&\le 4\varepsilon^{\frac{1}{3}}\tau NC^2|\mathcal{X}(n)-\mathcal{X}(\bar{n})|_\varepsilon+\frac{2\rho C}{1+\nu/2}|n-\bar{n}|_0,
\end{aligned}
$$

which implies the second inequality in (2.22) at once. This completes the proof of Lemma 2.2. □

Define $\mathcal{V}(n)=V(\mathcal{X}(n))$. Then from Lemma 2.2, (2.19), and (2.20), we have $\mathcal{V}(n)'=V_1(\mathcal{X}(n))$, $\mathcal{V}(n)''=V_2(\mathcal{X}(n))$ and $\mathcal{V}(n)'''=V_3(\mathcal{X}(n))$, where $'=d/dz$, and the following:

Corollary 2.1. *For given positive number N, there exists an $\varepsilon_0=\varepsilon_0(N)>0$ such that if $0<\varepsilon<\varepsilon_0$ and $n\in Y_N$, then $v:=\mathcal{V}(n)$ satisfies (1.15), $\mathcal{V}(n)\in W_1$,*

$$
|\mathcal{V}(n)|_0+\varepsilon^{\frac{1}{3}}|\mathcal{V}(n)'|_0+\varepsilon^{\frac{2}{3}}|\mathcal{V}(n)''|_0+\varepsilon|\mathcal{V}(n)'''|_0\le\tilde{C}|n|_0, \tag{2.25}
$$

and

$$
\begin{aligned}
&|\mathcal{V}(n)-\mathcal{V}(\bar{n})|_0+\varepsilon^{\frac{1}{3}}|\mathcal{V}(n)'-\mathcal{V}(\bar{n})'|_0+\varepsilon^{\frac{2}{3}}|\mathcal{V}(n)''-\mathcal{V}(\bar{n})''|_0\\
&\quad+\varepsilon|\mathcal{V}(n)'''-\mathcal{V}(\bar{n})'''|_0\le\tilde{C}|n-\bar{n}|_0,
\end{aligned} \tag{2.26}
$$

for any $n,\bar{n}\in Y_N$, where $\tilde{C}>0$ is a constant independent of ε and $n\in Y_N$.

Next we discuss the equation (1.16) for n. Let

$$
H_1(n,v'')=\frac{\alpha\gamma(1-\varepsilon\tau\rho_1v'')n}{(1+\gamma n)^2},\qquad H_2(n,v''')=\frac{\varepsilon\alpha\tau\rho_1v'''}{1+\gamma n}. \tag{2.27}
$$

If $1+H_1>0$, then the equation (1.16) is equivalent to

$$
m'=n(1-n),\qquad \varepsilon n'=\frac{1}{1+H_1}(n-m-\varepsilon nH_2). \tag{2.28}
$$

Let $m=n_0+m_1$ and $n=m+n_1$, where $n_0=\frac{e^z}{1+e^z}$. We have

$$
m_1' = (1-2n_0)m_1-m_1^2+(1-2n_0-2m_1)n_1-n_1^2, \tag{2.29}
$$

$$
\varepsilon n_1' = \frac{n_1}{1+H_1}-\frac{\varepsilon nH_2}{1+H_1}-\varepsilon m', \tag{2.30}
$$

where in the right-hand side of (2.30), m' should be substituted by

$$
m'=n_0'+m_1'=n_0(1-n_0)+(1-2n_0)m_1-m_1^2+(1-2n_0-2m_1)n_1-n_1^2. \tag{2.31}
$$

Lemma 2.3. *For each sufficiently small $\sigma > 0$, there exists a positive number σ_1 with $\sigma_1 < \tilde{D}\sigma$, where $\tilde{D}_1 > 0$ is a constant independent of σ, such that for each $n_1 \in Z_\sigma$ there is a unique $m_1 := \mathcal{M}_1(n_1) \in Z_{\sigma_1}$ satisfying (2.29) and $\mathcal{M}_1(n_1)(0) = 0$. Furthermore, for any $n_1, \bar{n}_1 \in Z_\sigma$,*

$$|\mathcal{M}_1(n_1)|_0 \le D|n_1|_0, \qquad |\mathcal{M}_1(n_1) - \mathcal{M}_1(\bar{n}_1)|_0 \le D|n_1 - \bar{n}_1|_0, \tag{2.32}$$

where D is a constant independent of σ and $n_1 \in Z_\sigma$.

Proof. Given a small $\sigma > 0$ and $n_1 \in Z_\sigma$, let $\sigma_1 > 0$ be a small number which will be chosen later. We define a mapping $\mathcal{Q}$ on Z_{σ_1} by

$$\mathcal{Q}(m_1)(z) = \int_0^z e^{\int_s^z (1-2n_0(\eta))\,d\eta}[-m_1^2 + (1 - 2n_0 - 2m_1)n_1 - n_1^2]\,ds. \tag{2.33}$$

We first show that $\mathcal{Q}$ maps Z_{σ_1} into itself. Since $\lim_{z\to-\infty}(1 - 2n_0(z)) = 1$ and $\lim_{z\to\infty}(1 - 2n_0(z)) = -1$, it follows that there is a constant $D_1 > 0$ such that

$$\left|\int_0^z e^{\int_s^z (1-2n_0(\eta))\,d\eta}\,ds\right| \le D_1, \quad \text{for } z \in (-\infty, \infty).$$

Hence, using $|1 - 2n_0|_0 \le 1$, we have $|\mathcal{Q}(m_1)|_0 \le D_1(\sigma_1^2 + (1 + 2\sigma_1)\sigma + \sigma^2) = \sigma_1$ if we take

$$\sigma_1 = \frac{1}{2}\left\{\Big(\frac{1}{D_1} - 2\sigma\Big) - \sqrt{\Big(\frac{1}{D_1} - 2\sigma\Big)^2 - 4(\sigma + \sigma^2)}\,\right\} \sim D_1\sigma, \quad \text{as } \sigma \to 0.$$

Therefore, σ_1 is well defined if σ is taken sufficiently small, and $|\mathcal{Q}(m_1)|_0 \le \sigma_1$ if $|m_1|_0 \le \sigma_1$. It is easy to verify that $\lim_{z\to\pm\infty} \mathcal{Q}(m_1)(z) = 0$. This shows that $\mathcal{Q}$ maps Z_{σ_1} into itself.

It follows from (2.33) that, for any m_1 and $\bar{m}_1$ in Z_{σ_1}

$$|\mathcal{Q}(m_1) - \mathcal{Q}(\bar{m}_1)|_0 \le D_1(2\sigma_1 + 2\sigma)|m_1 - \bar{m}_1|_0.$$

Hence, if we take σ small enough so that $2D_1(\sigma_1 + \sigma) < 1$, then $\mathcal{Q}$ is a contraction on Z_{σ_1} and therefore has a unique fixed point $m_1 := \mathcal{M}(n_1) \in Z_{\sigma_1}$. Clearly, $\mathcal{M}_1(n_1)$ satisfies (2.29) and $\mathcal{M}_1(n_1)(0) = 0$. This shows the first part of the lemma.

It remains to show (2.32). It follows from (2.33) that, for any n_1 and $\bar{n}_1$ in Z_σ,

$$\begin{aligned}
|\mathcal{M}_1(n_1) - \mathcal{M}_1(\bar{n}_1)|_0 \;\le\;& D_1[2\sigma_1|\mathcal{M}_1(n_1) - \mathcal{M}_1(\bar{n}_1)|_0 + |1 - 2n_0(s)||n_1 - \bar{n}_1|_0 \\
&\quad + |n_1|_0|\mathcal{M}_1(n_1) - \mathcal{M}_1(\bar{n}_1)|_0 + |\mathcal{M}_1(\bar{n}_1)|_0|n_1 - \bar{n}_1|_0 \\
&\quad + 2\sigma||n_1 - \bar{n}_1|_0] \\
\le\;& D_1(2\sigma_1 + \sigma)|\mathcal{M}_1(n_1) - \mathcal{M}_1(\bar{n}_1)|_0 \\
&\quad + D_1(1 + \sigma_1 + 2\sigma)|n_1 - \bar{n}_1|_0,
\end{aligned}$$

from which the second inequality in (2.32) follows immediately if σ is taken sufficiently small. Setting $\bar{n}_1 = 0$ in this inequality yields the first inequality in (2.32). This completes the proof of Lemma 2.3. □

Lemma 2.4. *There is a positive number σ_0 such that for each $0 < \sigma < \sigma_0$, if ε is sufficiently small, then there is a unique $n_1 \in Z_\sigma$ satisfying the equation (2.30), in which m, m', n, $H_1(n, v'')$, $H_2(n, v''')$ are replaced by $\mathcal{M}(n_1) := n_0 + \mathcal{M}_1(n_1)$, $\mathcal{M}(n_1)'$, $\mathcal{N}(n_1) := \mathcal{M}(n_1) + n_1$, $\mathcal{H}_1(n_1) := H_1(\mathcal{N}(n_1), \mathcal{V}(\mathcal{N}(n_1))'')$, and $\mathcal{H}_2(n_1) := H_2(\mathcal{N}(n_1), \mathcal{V}(\mathcal{N}(n_1))''')$ respectively. Here $\mathcal{V}$ is given in Corollary 2.1 with $N = 2$. Furthermore, $|n_1|_0 + \varepsilon|n_1'|_0 \le M\varepsilon$, where $M > 0$ is a constant independent of ε.*

Proof. Given a sufficiently small $\sigma > 0$, we define a mapping $\mathcal{R}$ on Z_σ by

$$\mathcal{R}(n_1)(z) = \int_z^\infty e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(n_1)(\eta)\,d\eta}[\mathcal{N}(n_1)(s)\mathcal{H}_2(n_1)(s)\mathcal{H}_3(n_1)(s) + \mathcal{M}(n_1)'(s)]\,ds, \tag{2.34}$$

where $\mathcal{H}_3(n_1) := 1/(1 + \mathcal{H}_1(n_1))$. We first show that $\mathcal{R}$ maps Z_σ into itself. To do so, we need to do some preliminary work.

First, from Lemma 2.3 we have $-(D+1)\sigma \le \mathcal{N}(n_1) < 1 + (D+1)\sigma < 2$, and so

$$\frac{1}{2} \le 1 - \gamma(D+1)\sigma \le 1 + \gamma\mathcal{N}(n_1) < 1 + 2\gamma, \tag{2.35}$$

if σ is taken sufficiently small. Recall from (2.25) that $\varepsilon^{\frac{2}{3}}|\mathcal{V}(\mathcal{N}(n_1))''|_0 \le \tilde{C}N$. It follows that

$$-\alpha\gamma(1 + \tilde{C}N\varepsilon^{\frac{1}{3}})(D+1)\sigma \le \alpha\gamma[1 - \varepsilon\tau\rho_1\mathcal{V}(\mathcal{N}(n_1))'']\mathcal{N}(n_1) \le 2\alpha\gamma(1 + \tilde{C}N\varepsilon^{\frac{1}{3}}),$$

and hence from (2.35)

$$-4\alpha\gamma(1 + \tilde{C}N\varepsilon^{\frac{1}{3}})(D+1)\sigma \le \mathcal{H}_1(n_1) \le 8\alpha\gamma(1 + \tilde{C}N\varepsilon^{\frac{1}{3}}).$$

Thus, if we take σ and ε small,

$$\frac{1}{2} \le 1 + \mathcal{H}_1(n_1) \le 1 + 9\alpha\gamma, \quad \text{so} \quad \kappa := \frac{1}{1 + 9\alpha\gamma} \le \mathcal{H}_3(n_1) \le 2, \tag{2.36}$$

and so,

$$e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(n_1)(\eta)\,d\eta} \le e^{\frac{\kappa}{\varepsilon}(z-s)}, \quad \text{for } s \ge z. \tag{2.37}$$

Again, from (2.25) we have $\varepsilon|\mathcal{V}(\mathcal{N}(n_1))'''|_0 \le C^*N$ and so $|\mathcal{H}_2(n_1)|_0 \le 2\alpha\tau\rho_1\tilde{C}N$. Clearly from $\mathcal{M}(n_1)' = n_0' + \mathcal{M}_1(n_1)'$ and (2.29) we see that $\mathcal{N}(n_1)'$ is bounded with a bound independent of n_1 and ε. Therefore, there exists a constant $M_1 > 0$ independent of n_1 and ε such that $|\mathcal{N}(n_1)\mathcal{H}_2(n_1)\mathcal{H}_3(n_1)|_0 + |\mathcal{M}(n_1)'|_0 \le M_1$. Hence from (2.34) we have

$$|\mathcal{R}(n_1)(z)| \le M_1 \int_z^\infty e^{\frac{\kappa}{\varepsilon}(z-s)}\,ds = \frac{M_1}{\kappa}\varepsilon < \sigma, \tag{2.38}$$

provided that ε is sufficiently small. One can show easily that $\lim_{z\to\pm\infty} \mathcal{R}(n_1)(z) = 0$. This shows that $\mathcal{R}$ maps Z_σ into itself.

The rest of the proof is to show that $\mathcal{R}$ is a contraction on Z_σ. Given $n_1 \in Z_\sigma$ and $\bar{n}_1 \in Z_\sigma$, it follows from (2.34) that

$$\begin{aligned}
\mathcal{R}(n_1)(z) - \mathcal{R}(\bar{n}_1)(z) &= \int_z^\infty e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(n_1)(\eta)\,d\eta}[\mathcal{H}_2(n_1)\mathcal{H}_3(n_1)\mathcal{N}(n_1) + \mathcal{M}(n_1)']\,ds \\
&\quad - \int_z^\infty e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(\bar{n}_1)(\eta)\,d\eta}[\mathcal{H}_2(\bar{n}_1)\mathcal{H}_3(\bar{n}_1)\mathcal{N}(\bar{n}_1) + \mathcal{M}(\bar{n}_1)']\,ds \\
&= \int_z^\infty \left(e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(n_1)(\eta)\,d\eta} - e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(\bar{n}_1)(\eta)\,d\eta}\right) \\
&\quad \times [\mathcal{H}_2(n_1)\mathcal{H}_3(n_1)\mathcal{N}(n_1) + \mathcal{M}(n_1)']\,ds \\
&\quad + \int_z^\infty e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(\bar{n}_1)(\eta)\,d\eta} \\
&\quad \times \{[\mathcal{H}_2(n_1)\mathcal{H}_3(n_1)\mathcal{N}(n_1) + \mathcal{M}(n_1)'] \\
&\qquad - [\mathcal{H}_2(\bar{n}_1)\mathcal{H}_3(\bar{n}_1)\mathcal{N}(\bar{n}_1) + \mathcal{M}(\bar{n}_1)']\}\,ds \\
&= I_1(z) + I_2(z).
\end{aligned}$$

We first estimate I_1. By the mean value theorem,

$$e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(n_1)(\eta)\,d\eta} - e^{\frac{1}{\varepsilon}\int_s^z \mathcal{H}_3(\bar{n}_1)(\eta)\,d\eta} = e^{\frac{1}{\varepsilon}K(s,z)}\frac{1}{\varepsilon}\int_s^z (\mathcal{H}_3(\bar{n}_1)(\eta) - \mathcal{H}_3(n_1)(\eta))\,d\eta,$$

where

$$K(s,z) = \zeta(s,z)\int_s^z \mathcal{H}_3(n_1)(\eta)\,d\eta + (1-\zeta(s,z))\int_s^z \mathcal{H}_3(\bar{n}_1)(\eta)\,d\eta,$$

and $0 < \zeta(s,z) < 1$. From (2.37) we get $e^{\frac{1}{\varepsilon}K(s,z)} \le e^{\frac{\kappa}{\varepsilon}(z-s)}$ for $s \ge z$, and from (2.36) we get

$$|\mathcal{H}_3(\bar{n}_1) - \mathcal{H}_3(n_1)|_0 \le \frac{|\mathcal{H}_1(n_1) - \mathcal{H}_1(\bar{n}_1)|_0}{(1+\mathcal{H}_1(\bar{n}_1))(1+\mathcal{H}(n_1))} \le 4|\mathcal{H}_1(n_1) - \mathcal{H}_1(\bar{n}_1)|_0. \quad (2.39)$$

From (2.25), (2.26), (2.36), and (2.32), we have for both $n = \mathcal{N}(n_1)$ and $n = \mathcal{N}(\bar{n}_1)$,

$$\begin{aligned}
&\frac{\partial H_1}{\partial n} = \alpha\gamma(1-\varepsilon\tau\rho_1 v'')\frac{1-\gamma n}{(1+\gamma n)^3} = O(1), \qquad \frac{\partial H_1}{\partial v''} = \frac{-\varepsilon\alpha\gamma\tau\rho_1 n}{(1+\gamma n)^2} = O(\varepsilon), \\
&|\mathcal{N}(n_1) - \mathcal{N}(\bar{n}_1)|_0 \le |n_1 - \bar{n}_1|_0 + |\mathcal{M}_1(n_1) - \mathcal{M}_1(\bar{n}_1)|_0 \le (1+D)|n_1 - \bar{n}_1|_0, \\
&\varepsilon^{\frac{2}{3}}|\mathcal{V}(\mathcal{N}(n_1))'' - \mathcal{V}(\mathcal{N}(\bar{n}_1))''|_0 \le \tilde{C}|\mathcal{N}(n_1) - \mathcal{N}(\bar{n}_1)|_0.
\end{aligned}$$

It again follows from the mean value theorem that there is a constant $M_2 > 0$ independent of $n_1 \in Z_\sigma$ and ε such that

$$\begin{aligned}
|\mathcal{H}_1(n_1) - \mathcal{H}_1(\bar{n}_1)|_0 &\le M_2\{|\mathcal{N}(n_1) - \mathcal{N}(\bar{n}_1)|_0 + \varepsilon|\mathcal{V}(\mathcal{N}(n_1))'' - \mathcal{V}(\mathcal{N}(\bar{n}_1))''|_0\} \\
&\le M_2(1+\tilde{C}\varepsilon^{\frac{1}{3}})|\mathcal{N}(n_1) - \mathcal{N}(\bar{n}_1)|_0 \le \frac{1}{4}M_3|n_1 - \bar{n}_1|_0,
\end{aligned}$$

where $\frac{1}{4}M_3 = 2M_2(1 + D)$ if ε is small. Hence, using (2.39), we get

$$\frac{1}{\varepsilon}\left|\int_s^z (\mathcal{H}_3(\bar{n}_1) - \mathcal{H}_3(n_1))\,d\eta\right| \le \frac{M_3}{\varepsilon}(s - z)|n_1 - \bar{n}_1|_0, \quad \text{for } s \ge z.$$

One can easily check that $|\mathcal{H}_2(n_1)\mathcal{H}_3(n_1)\mathcal{N}(n_1) + \mathcal{M}(n_1)'|_0 \le M_4$ for some constant $M_4 > 0$ independent of n_1 and ε. It follows that for $z \in (-\infty, \infty)$,

$$\begin{aligned} |I_1(z)| &\le M_3M_4|n_1 - \bar{n}_1|_0\frac{1}{\varepsilon}\int_z^\infty e^{-\frac{\kappa}{\varepsilon}(s-z)}(s - z)\,ds \\ &= M_3M_4|n_1 - \bar{n}_1|_0\frac{1}{\varepsilon}\int_0^\infty e^{-\frac{\kappa}{\varepsilon}\eta}\eta\,d\eta = M_3M_4\frac{\varepsilon}{\kappa^2}|n_1 - \bar{n}_1|_0. \end{aligned}$$

Next we find an estimate for $I_2(z)$. Note that $\frac{\partial H_2}{\partial v'''} = \frac{\varepsilon\alpha\tau\rho_1}{1+\gamma n}$ and $\frac{\partial H_2}{\partial n} = -\frac{\varepsilon\alpha\tau\rho_1\gamma v'''}{(1+\gamma n)^2}$. Thus, for both n_1 and $\bar{n}_1$ it follows from (2.25) that $|\frac{\partial H_2}{\partial v'''}| \le 2\alpha\tau\rho_1\varepsilon$, and $|\frac{\partial H_2}{\partial n}| \le 4\alpha\tau\rho_1\tilde{C}$. Then, it is easy to show by the mean value theorem and (2.26) that there is a constant $M_5 > 0$ independent of n_1 and ε such that

$$|\mathcal{H}_2(n_1)\mathcal{H}_3(n_1)\mathcal{N}(n_1) - \mathcal{H}_2(\bar{n}_1)\mathcal{H}_3(\bar{n}_1)\mathcal{N}(\bar{n}_1)|_0 \le M_5|n_1 - \bar{n}_1|_0.$$

Similarly, by the right-hand equation of $\mathcal{M}_1'$ in (2.29) and Lemma 2.3 we obtain

$$|\mathcal{M}(n_1)' - \mathcal{M}(\bar{n}_1)'|_0 = |\mathcal{M}_1(n_1)' - \mathcal{M}_1(\bar{n}_1)'|_0 \le M_6|n_1 - \bar{n}_1|_0,$$

where $M_6 > 0$ is independent of n_1 and ε. Hence, using (2.37) and $\int_z^\infty e^{-\frac{\kappa}{\varepsilon}(s-z)}\,ds = \frac{\varepsilon}{\kappa}$, we get $|I_2(z)| \le \frac{1}{\kappa}M_5M_6\varepsilon|n_1 - \bar{n}_1|_0$ for $z \in (-\infty, \infty)$. Therefore,

$$|\mathcal{R}(n_1) - \mathcal{R}(\bar{n}_1)|_0 \le |I_1|_0 + |I_2|_0 \le \left(\frac{1}{\kappa^2}M_3M_4 + \frac{1}{\kappa}M_5M_6\right)\varepsilon|n_1 - \bar{n}_1|_0,$$

which implies that $\mathcal{R}$ is a contraction if ε is sufficiently small. Thus, $\mathcal{R}$ has a unique fixed point $n_1 \in Z_\sigma$. Furthermore, from (2.38) we see that $|n_1|_0 \le \frac{1}{\kappa}M_1\varepsilon$. It then follows from (2.30) that $|n_1'|_0 \le M$ for some constant $M > 0$ independent of ε. This completes the proof of Lemma 2.4. □

Proof of Theorem 1.2. Let n_1 be the solution of (2.30) given in Lemma 2.4. It follows from the above lemmas that $(v, n) := (\mathcal{V}(\mathcal{N}(n_1)), \mathcal{N}(n_1))$ gives a solution to (1.15)–(1.17). The rest of the proof is devoted to show (1.18).

It follows from Corollary 2.1 and Lemmas 2.3 and 2.4 that, for sufficiently small ε,

$$|v|_0 + \varepsilon^{\frac{1}{3}}|v'|_0 + \varepsilon^{\frac{2}{3}}|v''|_0 + \varepsilon|v'''|_0 \le C, \qquad |n - n_0|_0 + \varepsilon|n'|_0 \le C\varepsilon. \tag{2.40}$$

Now we show that for sufficiently small ε,

$$|v'|_0 + \varepsilon^{\frac{1}{3}}|v''|_0 + \varepsilon^{\frac{2}{3}}|v'''|_0 + \varepsilon|v^{(4)}|_0 \le C, \qquad \varepsilon^2|v^{(5)}|_0 \le C. \tag{2.41}$$

Differentiating the equation (1.15) with respect to z, we get

$$\beta\varepsilon^2(v')^{(4)} - \mu\varepsilon(v')^{(3)} - \varepsilon(v')'' + \rho(v') = f_1 := \rho\frac{d}{dz}\left\{\frac{n}{1 + \nu(1 - \varepsilon\tau\rho_1 v'')}\right\}. \tag{2.42}$$

From (2.40) we see that $|f_1|_0 \le C$ if ε is sufficiently small. Then applying Lemma 2.1 to (2.42) yields the first inequality in (2.41). The second inequality in (2.41) follows from (2.42) and the first inequality in (2.41).

We show that $|n''|_0 \le C$ for sufficiently small ε. First from (1.16) it follows that $\lim_{z\to\pm\infty} n''(z) = 0$. Therefore, $|n''|$ reaches its maximum at a finite z where $n''' = 0$. Differentiating the equation (1.16) gives

$$\varepsilon n''' - n'' + \frac{d}{dz} n(1-n) = \varepsilon\alpha \frac{d^2}{dz^2}\left\{ n \frac{d}{dz}\left(\frac{1-\varepsilon\tau\rho_1 v''}{1+\gamma n} \right) \right\}. \tag{2.43}$$

Setting $n''' = 0$ and using (2.40) we obtain $|n''|_0 \le C$.

By continuing to differentiate the equations (2.42) and (2.43), we can show in a similar manner that, for any positive integer j,

$$|v^{(j)}|_0 + \varepsilon^{\frac{1}{3}}|v^{(j+1)}|_0 + \varepsilon^{\frac{2}{3}}|v^{(j+2)}|_0 + \varepsilon|v^{(j+3)}|_0 + \varepsilon^2|v^{(j+4)}|_0 \le C_j, \qquad |n^{(j)}|_0 \le C_j, \tag{2.44}$$

where C_j is a positive number independent of ε and j.

We now use (2.44) to show the second inequality in (1.18). Note that

$$(n-n_0)' - (1-n-n_0)(n-n_0) = \varepsilon n'' - \varepsilon\alpha \frac{d^2}{dz^2}\left\{ n \frac{d}{dz}\left(\frac{1-\varepsilon\tau\rho_1 v''}{1+\gamma n} \right) \right\}. \tag{2.45}$$

Using (2.44) and the second inequality in (2.40) we obtain $|n' - n_0'|_0 \le C\varepsilon$. Then we continue to differentiate (2.45) to get $|n^{(j)} - n_0^{(j)}|_0 \le C_j\varepsilon$ for any positive integer j.

Similarly, we note that

$$\begin{aligned} \rho(v-v_0) \;=\; & -\beta\varepsilon^2 v'''' + \mu\varepsilon v''' + \varepsilon v'' \\ & + \frac{\rho(n-n_0)}{1+\nu(1-\varepsilon\tau\rho_1 v'')} + \frac{\varepsilon\tau v'' n_0}{(1+\nu)(1+\nu(1-\varepsilon\tau\rho_1 v''))}, \end{aligned} \tag{2.46}$$

where $v_0 := \frac{n_0}{1+\nu}$. From (2.44) and the estimate for $|n-n_0|_0$ we get that $|v-v_0|_0 \le C\varepsilon$. Then by differentiating the equation (2.46) and using (2.44) and estimates for $|n^{(j)} - n_0^{(j)}|_0$ obtained above, we get $|v^{(j)} - v_0^{(j)}|_0 \le C_j\varepsilon$ for any positive integer j. This shows (1.18), thereby completing the proof of Theorem 1.2. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $(v_\varepsilon, n_\varepsilon)$ be a solution of (1.15)–(1.17) given by Theorem 1.2. Let $\theta_\varepsilon = v_\varepsilon''$. Then $(\theta, n) := (\theta_\varepsilon, n_\varepsilon)$ is a solution to (1.1)–(1.3) which satisfies (1.10). This shows the existence of $(\theta_\varepsilon, n_\varepsilon)$ satisfying (i) in Theorem 1.1.

It remains to show $n > 0$ on $(-\infty, \infty)$ and (ii) of Theorem 1.1. We first observe from (1.10) that for sufficiently small ε, $1 + \gamma n > \frac{1}{2}$, $1 + G_1(\theta, n) > \frac{1}{2}$ and $G_2(\theta) > 1 + \frac{1}{2}\nu$ on $(-\infty, \infty)$, where

$$G_1(\theta, n) := \frac{\alpha\gamma(1-\varepsilon\tau\rho_1\theta)n}{(1+\gamma n)^2}, \qquad G_2(\theta) := 1 + \nu(1-\varepsilon\tau\rho_1\theta).$$

Let $\theta_1 = \theta, \theta_2 = \theta', \theta_3 = \theta'', \theta_4 = \theta''', n_1 = n, n_2 = n'$, and $w = (\theta_1, \theta_2, \theta_3, \theta_4, n_1, n_2)$. Define

$$\begin{aligned} G(w) &= \frac{1}{\varepsilon(1 + G_1(\theta_1, n_1))} \left\{ -n_1(1-n_1) + n_2 - \frac{\varepsilon^2\alpha\tau\rho_1(\theta_3 n_1 + \theta_2 n_2)}{1+\gamma n_1} \right. \\ &\qquad + \frac{2\varepsilon^2\alpha\tau\rho_1\gamma\theta_2 n_1 n_2}{(1+\gamma n_1)^2} - \frac{\varepsilon\alpha\gamma(1-\varepsilon\tau\rho_1\theta_1)n_2^2}{(1+\gamma n_1)^2} \\ &\qquad \left. + \frac{2\varepsilon\alpha\gamma^2(1-\varepsilon\tau\rho_1\theta_1)n_1 n_2^2}{(1+\gamma n_1)^3} \right\}, \\ \bar{f}(w) &= \frac{\rho G(w)}{G_2(\theta_1)} + \frac{\varepsilon\nu\tau(2\theta_2 n_2 + \theta_3 n_1)}{G_2(\theta_1)^2} + \frac{2\varepsilon^2\nu^2\tau^2\rho_1\theta_2^2 n_1}{G_2(\theta_1)^3}, \\ H(w) &= \frac{1}{\beta\varepsilon^2}\{-\rho\theta_1 + \varepsilon\theta_3 + \mu\varepsilon\theta_4 + \bar{f}(w)\}. \end{aligned}$$

It follows that (1.1)–(1.3) is equivalent to the first order system for w:

$$\theta_1' = \theta_2, \qquad \theta_2' = \theta_3, \qquad \theta_3' = \theta_4, \qquad \theta_4' = H(w), \qquad n_1' = n_2, \qquad n_2' = G(w). \tag{3.1}$$

It is easy to verify that $w_0 = (0, 0, 0, 0, 0, 0)$ and $w_1 = (0, 0, 0, 0, 1, 0)$ are the only equilibria of (3.1). In order to study the stability of (3.1) at w_0 and w_1, we need to find the eigenvalues for its corresponding linearized systems at w_0 and w_1 respectively. Note that the characteristic equations for those systems are of the form:

$$\begin{aligned} &\left(\lambda^2 - \frac{\partial G}{\partial n_2}\lambda - \frac{\partial G}{\partial n_1}\right)\left(\lambda^4 - \frac{\partial H}{\partial \theta_4}\lambda^3 - \frac{\partial H}{\partial \theta_3}\lambda^2 - \frac{\partial H}{\partial \theta_2}\lambda - \frac{\partial H}{\partial \theta_1}\right) \\ &\qquad = \left(\frac{\partial H}{\partial \theta_6}\lambda - \frac{\partial H}{\partial \theta_5}\right)\left(\frac{\partial G}{\partial \theta_4}\lambda^3 - \frac{\partial G}{\partial \theta_3}\lambda^2 + \frac{\partial G}{\partial \theta_2}\lambda - \frac{\partial G}{\partial \theta_1}\right), \end{aligned} \tag{3.2}$$

where the partial derivatives for G and H are evaluated at w_0 and w_1 respectively. We have, at $w = w_0$,

$$\begin{aligned} &\frac{\partial G}{\partial \theta_1} = \frac{\partial G}{\partial \theta_2} = \frac{\partial G}{\partial \theta_3} = \frac{\partial G}{\partial \theta_4} = 0, \qquad \frac{\partial G}{\partial n_1} = -\frac{1}{\varepsilon}, \qquad \frac{\partial G}{\partial n_2} = \frac{1}{\varepsilon}, \\ &\frac{\partial H}{\partial \theta_1} = -\frac{\rho}{\beta\varepsilon^2}, \qquad \frac{\partial H}{\partial \theta_2} = 0, \qquad \frac{\partial H}{\partial \theta_3} = \frac{1}{\beta\varepsilon}, \qquad \frac{\partial H}{\partial \theta_4} = \frac{\mu}{\beta\varepsilon}, \\ &\frac{\partial H}{\partial n_1} = -\frac{\partial H}{\partial n_2} = -\frac{\rho}{\beta\varepsilon^3(1+\nu)}, \end{aligned}$$

and, at $w = w_1$,

$$\begin{aligned} &\frac{\partial G}{\partial \theta_1} = \frac{\partial G}{\partial \theta_2} = \frac{\partial G}{\partial \theta_4} = 0, \qquad \frac{\partial G}{\partial \theta_3} = -\frac{\varepsilon\alpha\tau\rho_1 L}{1+\gamma}, \qquad \frac{\partial G}{\partial n_1} = \frac{\partial G}{\partial n_2} = \frac{L}{\varepsilon}, \\ &\frac{\partial H}{\partial \theta_1} = -\frac{\rho}{\beta\varepsilon^2}, \qquad \frac{\partial H}{\partial \theta_2} = 0, \qquad \frac{\partial H}{\partial \theta_3} = \frac{\zeta}{\beta\varepsilon}, \qquad \frac{\partial H}{\partial \theta_4} = \frac{\mu}{\beta\varepsilon}, \\ &\frac{\partial H}{\partial n_1} = \frac{\partial H}{\partial n_2} = \frac{\rho L}{\beta\varepsilon^3(1+\nu)}, \end{aligned}$$

where

$$L = \frac{(1+\gamma)^2}{(1+\gamma)^2 + \alpha\gamma}, \qquad \zeta = 1 + \frac{\tau\nu}{(1+\nu)^2} - \frac{\alpha\tau L}{(1+\gamma)(1+\nu)}.$$

We remark that the last term in the definition of ζ was missing in [3]. Therefore, it follows from (3.2) that the characteristic equations at $w = w_0$ and $w = w_1$ are respectively

$$\left(\lambda^2 - \frac{1}{\varepsilon}\lambda + \frac{1}{\varepsilon}\right) \cdot (\beta\varepsilon^2\lambda^4 - \mu\varepsilon\lambda^3 - \varepsilon\lambda^2 + \rho) = 0, \tag{3.3}$$

and

$$\left(\lambda^2 - \frac{L}{\varepsilon}\lambda - \frac{L}{\varepsilon}\right) \cdot (\beta\varepsilon^2\lambda^4 - \mu\varepsilon\lambda^3 - \zeta\varepsilon\lambda^2 + \rho) = -\frac{\alpha\tau L^2}{(1+\gamma)(1+\nu)}\lambda^2(\lambda+1). \tag{3.4}$$

It follows from Lemma 5.2 that for sufficiently small ε, (3.3) has four positive solutions: $\lambda_{03} = \frac{1}{2\varepsilon}(1 - \sqrt{1-4\varepsilon}) \sim 1$, $\lambda_{04} \sim \sqrt[3]{\rho/\mu\varepsilon}$, $\lambda_{05} \sim \mu/\beta\varepsilon$, and $\lambda_{06} = \frac{1}{2\varepsilon}(1 + \sqrt{1-4\varepsilon}) \sim 1/\varepsilon$. An easy calculation shows that the corresponding eigenvectors are: $(0, 0, 0, 0, 1, \lambda_{03})^t$, $(1, \lambda_{0j}, \lambda_{0j}^2, \lambda_{0j}^3, 0, 0)^t$ $(j = 4, 5)$, and $(0, 0, 0, 0, 1, \lambda_{06})^t$. Similarly, it follows from Lemma 5.1 with $k = \alpha\tau L^2/(1+\gamma)(1+\nu)$ that (3.4) has two complex conjugate solutions λ_{11} and λ_{12} with $a_{11} := \Re(\lambda_{11}) = \Re(\lambda_{12}) \sim -\frac{1}{2}\sqrt[3]{\rho/\mu\varepsilon}$ and $b_{11} := \Im(\lambda_{12}) = \Im(\lambda_{12}) \sim \frac{\sqrt{3}}{2}\sqrt[3]{\rho/\mu\varepsilon}$ and one negative solution $\lambda_{13} \sim -1$. It is easily calculated that the corresponding eigenvectors for λ_{1j} $(j = 2, 3)$ are $(d_j, d_j\lambda_{1j}, d_j\lambda_{1j}^2, d_j\lambda_{1j}^3, 1, \lambda_{1j})^t$ with d_j defined in Theorem 1.1. It is easy to show that $d_j \neq 0$ $(j = 2, 3)$. Then, (1.13) follows directly from the stable manifold theorem (see [1], [6], [7]). Applying the stable manifold theorem at $w = w_0$ we have, as $z \to -\infty$,

$$\begin{pmatrix} \theta_\varepsilon(z) \\ \theta'_\varepsilon(z) \\ \theta''_\varepsilon(z) \\ \theta'''_\varepsilon(z) \\ n_\varepsilon(z) \\ n'_\varepsilon(z) \end{pmatrix} \sim c_{03} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \lambda_{03} \end{pmatrix} e^{\lambda_{03} z} + c_{04} \begin{pmatrix} 1 \\ \lambda_{04} \\ \lambda_{04}^2 \\ \lambda_{04}^3 \\ 0 \\ 0 \end{pmatrix} e^{\lambda_{04} z} + c_{05} \begin{pmatrix} 1 \\ \lambda_{05} \\ \lambda_{05}^2 \\ \lambda_{05}^3 \\ 0 \\ 0 \end{pmatrix} e^{\lambda_{05} z}$$
$$+ c_{06} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \lambda_{06} \end{pmatrix} e^{\lambda_{06} z}, \tag{3.5}$$

where c_{0j} $(j = 3, 4, 5, 6)$ are constants. Since $\lambda_{06} > \lambda_{03}$, (1.11) follows at once from (3.5) after we show below that $c_{03} > 0$. This proves (ii) of Theorem 1.1.

We now show that if ε is sufficiently small, then $c_{03} > 0$ and $n(z) > 0$ for all $z \in (-\infty, \infty)$. Since $n = n_0 + O(\varepsilon)$ and $n' = n'_0 + O(\varepsilon)$, we see $n > 0$ on $[0, \infty)$. We only need to show that $n > 0$ on $(-\infty, 0)$. To do so, we define

$$\bar{z} = \inf\{z \in (-\infty, 0) : 0 < n' < 2n \quad \text{on } (z, 0)\}.$$

Since $n(0) = \frac{1}{2}$ and $n'(0) = \frac{1}{2} + O(\varepsilon)$, we see that $\bar{z}$ is well defined. We claim that $\bar{z} = -\infty$. Assume that this is false. Then by the definition of $\bar{z}$, either $n'(\bar{z}) = 0$ or $0 < n'(\bar{z}) = 2n(\bar{z})$. (Note that the equation $n'' = G(w)$ implies that $n(\bar{z}) = n'(\bar{z}) \neq 0$, or else it would yield $n = n' \equiv 0$.) If $n'(\bar{z}) = 0$, then $n''(\bar{z}) \geq 0$ and $0 < n(\bar{z}) < 1/2$. However, if ε is sufficiently small, then at $z = \bar{z}$,

$$\begin{aligned} n'' &= G(w) = \frac{1}{\varepsilon(1+G_1(\theta,n))}\left[-(1-n) - \frac{\varepsilon^2\alpha\tau\rho_1\theta''}{1+\gamma n}\right]n \\ &= \frac{1}{\varepsilon(1+G_1(\theta,n))}[-(1-n)+O(\varepsilon^2)]n < 0. \end{aligned}$$

(Note that $1 + G_1(\theta, n) > 1/2$.) This contradiction excludes that $n'(\bar{z}) = 0$. If $n'(\bar{z}) = 2n(\bar{z})$, then $n''(\bar{z}) \leq 2n'(\bar{z})$. However, if ε is sufficiently small, then, at $z = \bar{z}$, we have $0 < G_1(\theta, n) \leq \alpha\gamma$ and so

$$n'' = G(w) = \frac{n'}{\varepsilon(1+G_1(\theta,n))}\left[-\frac{1-n}{2} + 1 + O(\varepsilon)\right] > \frac{n'}{2\varepsilon(1+\alpha\gamma)} > 2n'.$$

This again gives a contradiction. Hence, we have $\bar{z} = -\infty$, and therefore, $0 < n' < 2n$ on $(-\infty, 0)$, which implies that $c_{03} > 0$ (for if $c_{03} < 0$, then by (3.5) we have $n < 0$ as z is sufficiently large negative; if $c_{03} = 0$, then again from (3.5) we have $n' \sim \lambda_{06} n \sim (1/\varepsilon)n > 2n$ if ε is small). The proof of Theorem 1.1 is complete. □

4. Conclusions

We have proved the existence of travelling wave solutions for a tissue interaction model for skin pattern formation. Our emphasis has been on the case when the travelling wave speeds c are large, and on obtaining the qualitative results independent of c. We also obtain for those wave solutions the asymptotic behavior that is consistent with the approximation obtained by perturbation analysis in [3]. The global uniqueness and the stability of travelling wave solutions for each fixed but sufficiently large c will be considered in future works.

5. Appendix

The following lemmas have been used in the previous sections.

Lemma 5.1. *Let $\beta > 0$, $\mu > 0$, $\rho > 0$, ζ, $L > 0$, and $k \neq 0$ be real numbers.*
(i) If $\mu \neq L\beta$, then for sufficiently small $\varepsilon > 0$, the equation

$$(\beta\varepsilon^2\lambda^4 - \mu\varepsilon\lambda^3 - \zeta\varepsilon\lambda^2 + \rho)\left(\lambda^2 - \frac{L}{\varepsilon}\lambda - \frac{L}{\varepsilon}\right) = -k\lambda^2(\lambda+1) \tag{5.1}$$

has two complex conjugate roots $\lambda_1 = a - ib$ *and* $\lambda_2 = a + ib$, *and four real roots* $\lambda_3 < 0 < \lambda_4 < \lambda_5 < \lambda_6$ *such that as* $\varepsilon \to 0$,

$$\begin{cases} a = -\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}(1 + O(\varepsilon^{1/3})), \\ b = \frac{\sqrt{3}}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}(1 + O(\varepsilon^{1/3})), \\ \lambda_3 = -1 + O(\varepsilon), \\ \lambda_4 = \sqrt[3]{\frac{\rho}{\mu\varepsilon}}(1 + O(\varepsilon^{1/3})), \\ \lambda_5 = \begin{cases} \frac{\mu}{\beta\varepsilon}(1 + O(\varepsilon)), & \text{if } \mu < L\beta, \\ \frac{L}{\varepsilon}(1 + O(\varepsilon)), & \text{if } \mu > L\beta, \end{cases} \\ \lambda_6 = \begin{cases} \frac{L}{\varepsilon}(1 + O(\varepsilon)), & \text{if } \mu < L\beta, \\ \frac{\mu}{\beta\varepsilon}(1 + O(\varepsilon)), & \text{if } \mu > L\beta. \end{cases} \end{cases} \tag{5.2}$$

(ii) Assume that $\mu = L\beta$. *If* $k < 0$, *then the assertions in (i) hold except that, as* $\varepsilon \to 0$,

$$\begin{cases} \lambda_5 = \frac{L}{\varepsilon} - \sqrt{\frac{-k}{\mu\varepsilon}}(1 + O(\sqrt{\varepsilon})), \\ \lambda_6 = \frac{L}{\varepsilon} + \sqrt{\frac{-k}{\mu\varepsilon}}(1 + O(\sqrt{\varepsilon})). \end{cases}$$

If $k > 0$, *then the assertions in (i) hold except that* λ_5 *and* λ_6 *are complex conjugate numbers and, as* $\varepsilon \to 0$,

$$\begin{cases} \lambda_5 = \frac{L}{\varepsilon}(1 + O(\varepsilon)) - i\sqrt{\frac{k}{\mu\varepsilon}}(1 + O(\sqrt{\varepsilon})), \\ \lambda_6 = \frac{L}{\varepsilon}(1 + O(\varepsilon)) + i\sqrt{\frac{k}{\mu\varepsilon}}(1 + O(\sqrt{\varepsilon})). \end{cases}$$

Proof. Rewrite the equation (5.1) as

$$\begin{aligned} p(\lambda) \;:=\; &\beta\varepsilon^3\lambda^6 - \varepsilon^2(\mu + L\beta)\lambda^5 + \varepsilon(L\mu - \zeta\varepsilon - L\beta\varepsilon)\lambda^4 \\ &+ \varepsilon(L\zeta + L\mu + k)\lambda^3 + \varepsilon(\rho + L\zeta + k)\lambda^2 - L\rho\lambda - L\rho = 0. \end{aligned} \tag{5.3}$$

We first show (i). By the assumption we have $\mu \neq L\beta$. In order to show the existence of λ_5 and λ_6, we note that, for sufficiently small ε and $\lambda \in [\min\{\mu/(2\beta\varepsilon), L/(2\varepsilon)\}, \infty)$,

$$p(\lambda) = \varepsilon\lambda^4[\beta\varepsilon^2\lambda^2 - \varepsilon(\mu + L\beta)\lambda + L\mu + O(\varepsilon)],$$

where the constant in $O(\varepsilon)$ is independent of λ in this interval. It follows from the intermediate value theorem that for sufficiently small ε, $p(\lambda) = 0$ has exactly two real roots in $[\min\{\mu/(2\beta\varepsilon), L/(2\varepsilon)\}, \infty)$, and, as $\varepsilon \to 0$,

$$\begin{aligned} \lambda \;&=\; \frac{1}{2\beta\varepsilon^2}\{\varepsilon(\mu + L\beta) \pm \sqrt{\varepsilon^2(\mu + L\beta)^2 - 4\beta\varepsilon^2[L\mu + O(\varepsilon)]}\} \\ &=\; \frac{1}{2\beta\varepsilon}\{(\mu + L\beta) \pm |\mu - L\beta| + O(\varepsilon)\}. \end{aligned}$$

Namely, if $\mu < L\beta$, then $\lambda_5(k) = \frac{\mu}{\beta\varepsilon}(1 + O(\varepsilon))$ and $\lambda_6(k) = \frac{L}{\varepsilon}(1 + O(\varepsilon))$; and, if $\mu > L\beta$, then $\lambda_5(k) = \frac{L}{\varepsilon}(1 + O(\varepsilon))$, and $\lambda_6(k) = \frac{\mu}{\beta\varepsilon}(1 + O(\varepsilon))$.

To show the existence of λ_4, we note that for sufficiently small ε and all $\lambda \in [\frac{1}{2}\sqrt[3]{\rho/\mu\varepsilon}\,,\ \frac{3}{2}\sqrt[3]{\rho/\mu\varepsilon}]$,

$$p(\lambda) = \lambda[L\mu\varepsilon\lambda^3 - L\rho + O(\varepsilon^{1/3})], \tag{5.4}$$

where the constant in $O(\varepsilon^{1/3})$ is independent of λ in this interval. It follows that $p(\lambda) = 0$ has exactly one solution λ_4 in this interval, and, as $\varepsilon \to 0$,

$$\lambda_4 = \frac{\sqrt[3]{L\rho(1 + O(\varepsilon^{1/3}))}}{\sqrt[3]{L\mu\varepsilon}} = \sqrt[3]{\frac{\rho}{\mu\varepsilon}}(1 + O(\varepsilon^{1/3})).$$

To show the existence of λ_3, we note that for sufficiently small ε and all $\lambda \in [-2, 0]$, $p(\lambda) = -L\rho\lambda - L\rho + O(\varepsilon)$. Hence, $p(\lambda) = 0$ has a unique solution $\lambda_3 = -1 + O(\varepsilon)$ as $\varepsilon \to 0$.

To show the existence of λ_2, we consider $p(\lambda)$ in the region on the complex plane:

$$\left|\lambda - \left(-\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}} + i\frac{\sqrt{3}}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}\right)\right| \le \frac{1}{5}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}.$$

In this region, (5.4) still holds. Then a simple application of Rouche's theorem yields that for sufficiently small ε, $p(\lambda) = 0$ has a unique solution in this region which is denoted by λ_2, and has the asymptotic formula as $\varepsilon \to 0$ (following from (5.4)),

$$\lambda_2 = \sqrt[3]{\frac{\rho}{\mu\varepsilon}}(1 + O(\varepsilon^{1/3}))e^{i(2\pi/3 + O(\varepsilon^{1/3}))} = \sqrt[3]{\frac{\rho}{\mu\varepsilon}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(1 + O(\varepsilon^{1/3})).$$

Finally, we define λ_1 to be the complex conjugate of λ_2 which gives the sixth root of $p(\lambda) = 0$. We thereby complete the proof of (i).

We now show (ii). We have $\mu = L\beta$. We note that the above proof for the existence of λ_j $(j = 1, 2, 3, 4)$ still holds in this case. We only need to show the existence of λ_5 and λ_6 and their asymptotic formulas. To do so, we restrict $p(\lambda)$ in the region on the complex plane: $\left|\lambda - \frac{L}{\varepsilon}\right| \le 3\sqrt{\frac{k}{\mu\varepsilon}}$, and we have

$$\begin{aligned} p(\lambda) &= \varepsilon\lambda^4\left[\beta(\varepsilon\lambda - L)^2 - \varepsilon(\zeta + \beta L) + (L\zeta + L\mu + k)\frac{1}{\lambda} + O(\varepsilon^2)\right] \\ &= \varepsilon\lambda^4\left[\beta(\varepsilon\lambda - L)^2 - \varepsilon(\zeta + \beta L) + (L\zeta + L\mu + k)\frac{\varepsilon}{L}(1 + O(\sqrt{\varepsilon})) + O(\varepsilon^2)\right] \\ &= \beta\varepsilon\lambda^4\left[(\varepsilon\lambda - L)^2 + \frac{k}{\mu}\varepsilon(1 + O(\sqrt{\varepsilon}))\right]. \end{aligned}$$

It then follows from Rouche's theorem that, for sufficiently small ε, $p(\lambda) = 0$ has exactly two solutions in this neighborhood, and if $k < 0$, these two roots are both real such that

$\lambda = \frac{L}{\varepsilon} \pm \sqrt{\frac{-k}{\mu\varepsilon}}(1 + O(\sqrt{\varepsilon}))$, and if $k > 0$, those two roots are complex conjugate such that

$$\begin{aligned}\lambda &= \frac{1}{\varepsilon}\left[L + \sqrt{\frac{k\varepsilon}{\mu}}(1 + O(\sqrt{\varepsilon}))e^{\pm i(\pi/2 + O(\sqrt{\varepsilon}))}\right] \\ &= \frac{L}{\varepsilon}(1 + O(\varepsilon)) \pm i\sqrt{\frac{k}{\mu\varepsilon}}(1 + O(\sqrt{\varepsilon})).\end{aligned}$$

This completes the proof of (ii), and thereby the proof of Lemma 5.1. □

Lemma 5.2. *Let $\beta > 0$, $\mu > 0$, $\rho > 0$ be constants. If $\varepsilon > 0$ is sufficiently small, then the equation $p(\lambda) := \beta\varepsilon^2\lambda^4 - \mu\varepsilon\lambda^3 - \varepsilon\lambda^2 + \rho = 0$ has two complex roots $\lambda_1 = a - ib$ and $\lambda_2 = a + ib$, and two real roots $0 < \lambda_3 < \lambda_4$ such that as $\varepsilon \to 0$,*

$$\begin{cases} a = -\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}(1 + O(\varepsilon^{1/3})), \\ b = \frac{\sqrt{3}}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}(1 + O(\varepsilon^{1/3})), \\ \lambda_3 = \sqrt[3]{\frac{\rho}{\mu\varepsilon}}(1 + O(\varepsilon^{1/3})), \\ \lambda_4 = \frac{\mu}{\beta\varepsilon}(1 + O(\varepsilon)). \end{cases} \tag{5.5}$$

Proof. The proof is carried out in a similar manner to that of Lemma 5.1. In $(\frac{\mu}{2\beta\varepsilon}, 2\frac{\mu}{\beta\varepsilon})$ and $(\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}, 2\sqrt[3]{\frac{\rho}{\mu\varepsilon}})$, we have $p(\lambda) = \varepsilon\lambda^3(\beta\varepsilon\lambda - \mu + O(\varepsilon))$ and $p(\lambda) = -\mu\varepsilon[\lambda^3 - \frac{\rho}{\mu\varepsilon}(1 + O(\varepsilon^{1/3}))]$ respectively, which yield the existence of λ_4 and λ_3 and their asymptotic formulas in (5.5). For λ in the region $|\lambda - (-\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}} + i\frac{\sqrt{3}}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}})| \le \frac{1}{5}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}$ on the complex plane, we have $p(\lambda) = -\mu\varepsilon[\lambda^3 - \frac{\rho}{\mu\varepsilon}(1 + O(\varepsilon^{1/3}))]$ for sufficiently small ε, which yields the existence of λ_2 and its asymptotic formula in (5.5), and we note that λ_1 is the conjugate of λ_2. This completes the proof of the lemma. □

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