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Travelling Wave Solutions in a Tissue Interaction Model for Skin Pattern Formation*

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We discuss the existence and the uniqueness of travelling wave solutions for a tissue interaction model on skin pattern formation proposed by Cruywagen and Murray. The geometric theory of singular perturbations is employed.

KEY WORDS: Tissue interaction model; travelling wave solutions; singular perturbations.

1. INTRODUCTION

The skin of vertebrates, as the largest organ of the body, forms many specialized structures, for example, hair, scales, feathers, and glands, which are distributed over the skin in highly ordered fashion. The mechanisms involved in the formation and distribution of these appendages are not well understood, and, various mathematical models have been proposed for the purpose of the understanding of these mechanisms (see [11] and references therein).

Vertebrate skin is composed of two layers—the epidermis and the dermis. There is sound biological evidence that skin organ formation typically occurs due to interaction between these two layers. Based on this fact, Cruywagen and Murray [3] proposed a tissue interaction model for vertebrate skin pattern morphogenesis by using a mechanochemical mechanism to describe epithelial sheet motion and a reaction-diffusion-chemotaxis mechanism to model the dermal cell movements. Tissue interaction is introduced by the morphogens produced separately in the dermis and the epithelium. Those

* Dedicated to Professor Victor A. Pliss on the occasion of his 70th birthday.

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morphogens diffuse across the basal lamina, which separates the epidermis and the dermis, and induce cell movements and deformation. The model consists of seven coupled nonlinear partial differential equations: four to describe the production, degradation, and diffusion of the chemicals within and between layers, two conservation equations for dermal and epidermal cell densities, and a force balance equation for modelling stress in the epithelium. While the full system is too complicated to render any useful mathematical analysis, a special case of the model in one space dimension was considered in [3–5] where the full model is reduced into a system of two partial differential equations, which, after non-dimensionalization, has the form:

$$\beta \frac{\partial^4\tilde{\theta}}{\partial x^4} - \mu \frac{\partial^3\tilde{\theta}}{\partial t\,\partial x^2} - \frac{\partial^2\tilde{\theta}}{\partial x^2} + \rho\tilde{\theta} = \frac{\partial^2}{\partial x^2}\left\{\frac{\tau\tilde{n}}{1+\nu(1-\tilde{\theta})}\right\}, \tag{1.1}$$

$$\frac{\partial^2\tilde{n}}{\partial x^2} - \frac{\partial\tilde{n}}{\partial t} + \tilde{n}(1-\tilde{n}) = \alpha \frac{\partial}{\partial x}\left\{\tilde{n}\frac{\partial}{\partial x}\left(\frac{1-\tilde{\theta}}{1+\gamma\tilde{n}}\right)\right\}, \tag{1.2}$$

where $\tilde{\theta}$ stands for the epithelial dilation, $\tilde{n}$ stands for the dermal cell density, and, β, μ, ρ, τ, ν, α, γ are positive constants. We refer the readers to [3–5] and the references therein for a detailed derivation of the model and its biological background.

As a natural biological object in tissue interactions, the phenomenon of travelling waves for the system (1.1)–(1.2) was first investigated by Cruywagen, Maini, and Murray [4]. The travelling wave fronts $(\tilde{\theta}(\tilde{z}), \tilde{n}(\tilde{z}))$ satisfy a system of ordinary differential equations in $\tilde{z}$ and a pair of boundary conditions at $\tilde{z} = \pm\infty$:

$$\beta \frac{d^4\tilde{\theta}}{d\tilde{z}^4} - \mu c \frac{d^3\tilde{\theta}}{d\tilde{z}^3} - \frac{d^2\tilde{\theta}}{d\tilde{z}^2} + \rho\tilde{\theta} = \tau \frac{d^2}{d\tilde{z}^2}\left\{\frac{\tilde{n}}{1+\nu(1-\tilde{\theta})}\right\}, \tag{1.3}$$

$$\frac{d^2\tilde{n}}{d\tilde{z}^2} - c\frac{d\tilde{n}}{d\tilde{z}} + \tilde{n}(1-\tilde{n}) = \alpha \frac{d}{d\tilde{z}}\left\{\tilde{n}\frac{d}{d\tilde{z}}\left(\frac{1-\tilde{\theta}}{1+\gamma\tilde{n}}\right)\right\}, \tag{1.4}$$

$$\lim_{z\to-\infty} (\tilde{\theta}, \tilde{n}) = (0, 0), \qquad \lim_{z\to\infty} (\tilde{\theta}, \tilde{n}) = (0, 1), \tag{1.5}$$

where $\tilde{z} = x + ct$ and $c > 0$ is a travelling wave speed. Note that if $\alpha = 0$, then (1.4) decouples from (1.3) and becomes the classical Fisher equation which is known to exhibit travelling wave solutions only with wave speed $c \geqslant 2$. Based on this observation and the local stability analysis at the equilibria of (1.3)–(1.4), it was conjectured in [4] that (1.3)–(1.5) admit solutions for sufficiently large c. This motivates the re-scalings:

$$\tilde{\theta}(\tilde{z}) = \frac{\tau}{\rho c^2}\theta(z), \qquad \tilde{n}(\tilde{z}) = n(z), \qquad \tilde{z} = cz, \qquad \varepsilon = \frac{1}{c^2},$$

which reduce the equations (1.3)–(1.5) into the following singularly perturbed system:

$$\beta\varepsilon^2\frac{d^4\theta}{dz^4}-\mu\varepsilon\frac{d^3\theta}{dz^3}-\varepsilon\frac{d^2\theta}{dz^2}+\rho\theta=\rho\frac{d^2}{dz^2}\left\{\frac{n}{1+\nu(1-\varepsilon\tau\rho_1\theta)}\right\}, \tag{1.6}$$

$$\varepsilon\frac{d^2n}{dz^2}-\frac{dn}{dz}+n(1-n)=\varepsilon\alpha\frac{d}{dz}\left\{n\frac{d}{dz}\left(\frac{1-\varepsilon\tau\rho_1\theta}{1+\gamma n}\right)\right\}, \tag{1.7}$$

$$\lim_{z\to-\infty}(\theta,n)=(0,0), \qquad \lim_{z\to\infty}(\theta,n)=(0,1), \tag{1.8}$$

where $\rho_1=1/\rho$ and ε is sufficiently small. By using regular series expansions of the form

$$\theta(z)=\theta_0(z)+\varepsilon\theta_1(z)+\cdots, \qquad n(z)=n_0(z)+\varepsilon n_1(z)+\cdots,$$

an approximation to the solutions of (1.6)–(1.8) is obtained in [4], provided that they exist. In particular, the $O(1)$ terms of the above approximation read

$$\theta_0=\frac{1}{1+\nu}\frac{d^2n_0}{dz^2}, \qquad \frac{dn_0}{dz}=n_0(1-n_0). \tag{1.9}$$

Hence with the initial condition $n_0(0)=\frac{1}{2}$,

$$n_0(z)=e^z/(1+e^z). \tag{1.10}$$

Based on the contraction mapping principle, a rigorous proof of the existence of solutions $(\theta_\varepsilon, n_\varepsilon)$ to (1.6)–(1.8) was recently obtained in [1]. However, it was unknown that whether the wave solutions obtained in [1] are biologically meaningful in the sense that the density n_ε should always stay between 0 and 1, and, the dilation θ_ε should not tend to 0 in an oscillatory manner as $z\to\pm\infty$. The non-oscillatory behavior of $(\theta_\varepsilon, n_\varepsilon)$ is also an important issue when the stability of $(\theta_\varepsilon, n_\varepsilon)$ is considered, because an oscillatory wave solution can be unstable even in the sense of weighted norms (see Chapter 5 in [8]).

In this paper, we will use the geometric theory of singular perturbations to give a new proof for the existence of $(\theta_\varepsilon, n_\varepsilon)$. Not only is the new proof much simpler than that in [1] but also it provides more physical and geometrical insight into the wave solutions. For instance, we will actually show that $0<n_\varepsilon<1$ on $(-\infty,\infty)$ and θ_ε is non-oscillatory as $z\to\pm\infty$. Using the geometric theory, we will also obtain a global uniqueness result for $\nu=0$ within the class of physical solutions.

The main result of the paper is the following:

Theorem. *Let θ_0 and n_0 be as in (1.9), (1.10) respectively. Then the following holds for ε sufficiently small.*

(i) There is a unique solution $(\theta_\varepsilon, n_\varepsilon)$ to (1.6)–(1.8) that satisfies $n_\varepsilon(0) = \frac{1}{2}$, $n'_\varepsilon > 0$ on $(-\infty, \infty)$, and

$$\left| \frac{d^j}{dz^j} (\theta_\varepsilon(z) - \theta_0(z)) \right| \leqslant C_j \varepsilon, \qquad \left| \frac{d^j}{dz^j} (n_\varepsilon(z) - n_0(z)) \right| \leqslant C_j \varepsilon, \tag{1.11}$$

for all $z \in (-\infty, \infty)$ and $j = 0, 1, \ldots$, where, for each j, $C_j > 0$ is a constant independent of ε. Moreover, $(\theta_\varepsilon, n_\varepsilon)$ has the following asymptotic behavior:

$$\begin{pmatrix} \theta_\varepsilon(z) \\ \theta'_\varepsilon(z) \\ \theta''_\varepsilon(z) \\ \theta'''_\varepsilon(z) \\ n_\varepsilon(z) \\ n'_\varepsilon(z) \end{pmatrix} \sim c_{1-} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \lambda_{1-} \end{pmatrix} e^{\lambda_{1-} z} + c_{2-} \begin{pmatrix} 1 \\ \lambda_{2-} \\ \lambda_{2-}^2 \\ \lambda_{2-}^3 \\ 0 \\ 0 \end{pmatrix} e^{\lambda_{2-} z} + c_{3-} \begin{pmatrix} 1 \\ \lambda_{3-} \\ \lambda_{3-}^2 \\ \lambda_{3-}^3 \\ 0 \\ 0 \end{pmatrix} e^{\lambda_{3-} z} \tag{1.12}$$

as $z \to -\infty$, and,

$$\begin{pmatrix} \theta_\varepsilon(z) \\ \theta'_\varepsilon(z) \\ \theta''_\varepsilon(z) \\ \theta'''_\varepsilon(z) \\ n_\varepsilon(z) - 1 \\ n'_\varepsilon(z) \end{pmatrix} \sim c_{1+} \begin{pmatrix} d_{1+} \\ d_{1+}\lambda_{1+} \\ d_{1+}\lambda_{1+}^2 \\ d_{1+}\lambda_{1+}^3 \\ 1 \\ \lambda_{1+} \end{pmatrix} e^{\lambda_{1+} z}, \tag{1.13}$$

as $z \to \infty$, where, c_{j-}, λ_{j-} $(j = 1, 2, 3)$, c_{1+}, d_{1+}, λ_{1+} are constants such that $c_{1-} > 0$, $|c_{2-}| + |c_{3-}| \neq 0$, $c_{1+} < 0$, and, as $\varepsilon \to 0$,

$$\lambda_{1-} \sim 1, \quad \lambda_{2-} \sim \sqrt[3]{\frac{\rho}{\mu\varepsilon}}, \quad \lambda_{3-} \sim \frac{\mu}{\beta\varepsilon}, \quad \lambda_{1+} \sim -1 + \frac{1}{L}\varepsilon + \frac{k - 2\rho}{\rho L^2}\varepsilon^2,$$

$$d_{1+} = \frac{\lambda_{1+}^2 - \frac{L\lambda_{1+}}{\varepsilon} - \frac{L}{\varepsilon}}{\lambda_{1+}^2} \sim -\frac{\alpha\tau L}{\rho(1+\gamma)(1+\tau)}\varepsilon, \tag{1.14}$$

where

$$L = \frac{(1+\gamma)^2}{(1+\gamma)^2 + \alpha\gamma}, \qquad k = \frac{\alpha\tau L^2}{(1+\gamma)(1+\nu)}.$$

Hence θ_ε is non-oscillatory as $z \to \pm\infty$.

(ii) *If $\nu = 0$, then the solution to (1.6)–(1.8) is globally unique in the class of physical solutions, that is, if $(\theta_\varepsilon, n_\varepsilon)$ and $(\tilde{\theta}_\varepsilon, \tilde{n}_\varepsilon)$ are two solutions of (1.6)–(1.8) with $0 < n_\varepsilon < 1$, $0 < \tilde{n}_\varepsilon < 1$ and $\tilde{n}_\varepsilon(0) = n(0)$, then $(\tilde{\theta}_\varepsilon, \tilde{n}_\varepsilon) = (\theta_\varepsilon, n_\varepsilon)$.*

The geometric theory of singular perturbations has proven to be a powerful tool in the study of the existence of connecting orbits in singularly perturbed systems (see [9] and the references therein). However, although solutions to (1.6)–(1.8) are connecting orbits for (1.6)–(1.7), their existence does not directly follow from the geometric theory due to the appearance of multiple time scales in our problem. A key idea in the proof of our main result is to apply the geometric theory of singular perturbations twice, in order to separate different fast time scales in our system. We refer the readers to [2, 6, 8–10, 12] and the references therein for general literature of the geometric theory of singular perturbations and its applications.

The rest of the paper is devoted to the proof of our main result. In Section 2, we reformulate the system (1.6)–(1.8) into an equivalent system. The problems of existence and uniqueness of solutions for the new system will be treated in Section 3 and Section 4 respectively. Our proof for the uniqueness of solutions is motivated by that of [7], along with the application of two lemmas proved in [1]. For the reader's convenience, we include these lemmas in the Appendix.

2. AN EQUIVALENT SYSTEM

Let $v := \int_{-\infty}^{z} \int_{-\infty}^{\xi} \theta(\eta)\, d\eta\, d\xi$. Then it is to be seen that the system (1.6)–(1.8) is equivalent to the following system:

$$\beta\varepsilon^2 \frac{d^4 v}{dz^4} - \mu\varepsilon \frac{d^3 v}{dz^3} - \varepsilon \frac{d^2 v}{dz^2} + \rho v = \frac{\rho n}{1 + \nu(1 - \varepsilon\tau\rho_1 v'')}, \tag{2.1}$$

$$\varepsilon \frac{d^2 n}{dz^2} - \frac{dn}{dz} + n(1-n) = \varepsilon\alpha \frac{d}{dz}\left\{ n \frac{d}{dz}\left(\frac{1 - \varepsilon\tau\rho_1 v''}{1 + \gamma n} \right) \right\}, \tag{2.2}$$

$$\lim_{z \to -\infty} (v, n) = (0, 0), \qquad \lim_{z \to \infty} (v, n) = \left(\frac{1}{1+\nu}, 1 \right). \tag{2.3}$$

Consequently, our main result stated in the previous section can be re-formulated with respect to the new system as follows.

Theorem 2.1. *Let $v_0 = n_0/(1+\nu)$ and ε be sufficiently small. Then the following holds for (2.1)–(2.3).*

(i) There exists a unique solution $(v_\varepsilon, n_\varepsilon)$ which satisfies $n_\varepsilon(0) = \frac{1}{2}$ and

$$\left|\frac{d^j}{dz^j}(v_\varepsilon(z) - v_0(z))\right| \leqslant C_j \varepsilon, \qquad \left|\frac{d^j}{dz^j}(n_\varepsilon(z) - n_0(z))\right| \leqslant C_j \varepsilon, \quad (2.4)$$

for all $z \in (-\infty, \infty)$ and $j = 0, 1, \ldots$, where, for each j, $C_j > 0$ is a constant independent of ε. Moreover, $n'_\varepsilon > 0$, $v_\varepsilon > 0$, $v'_\varepsilon > 0$ on $(-\infty, \infty)$, and

$$v_\varepsilon(z) = \frac{n_\varepsilon(z)}{1+\nu} + O(\varepsilon), \quad n_\varepsilon(z) = \frac{e^{(1+O(\varepsilon))\,z}}{1+e^{(1+O(\varepsilon))\,z}}, \qquad -\infty < z < \infty. \tag{2.5}$$

(ii) If $\nu = 0$, then the solution $(v_\varepsilon, n_\varepsilon)$ is globally unique within the class of physical solutions in the sense that whenever $(v_\varepsilon, n_\varepsilon)$ and $(\tilde{v}_\varepsilon, \tilde{n}_\varepsilon)$ are two solutions of (2.1)–(2.3) with $0 < n_\varepsilon < 1$, $0 < \tilde{n}_\varepsilon < 1$, and $\tilde{n}_\varepsilon(0) = n(0)$, then $(\tilde{v}_\varepsilon, \tilde{n}_\varepsilon) = (v_\varepsilon, n_\varepsilon)$.

Theorem 2.1 does imply our main result except the formulas (1.12) and (1.13). We note that a solution $(v_\varepsilon, n_\varepsilon)$ of (2.1)–(2.3) given in part (i) of Theorem 2.1 clearly yields a solution $(\theta_\varepsilon, n_\varepsilon) := (v''_\varepsilon, n_\varepsilon)$ of (1.6)–(1.8) which satisfies (1.11). Conversely, if (θ, n) is a solutions to (1.6)–(1.8), then (v, n), where $v = \int_{-\infty}^{z}\int_{-\infty}^{\xi} \theta(\eta)\, d\eta\, d\xi$, is a solution to (2.1)–(2.3). This assertion follows from the fact that $\lim_{z\to\infty} v = 1/(1+\nu)$, which follows from Lemma 5.1 with $f = \frac{\rho n}{1+\nu(1-\varepsilon\tau\rho_1\theta)}$. Thus, the uniqueness results in both parts of our main result follows from the corresponding parts in Theorem 2.1.

The formulas (1.12) and (1.13) follow from a local analysis at the equilibria of (1.6)–(1.7) similar to [1]. The fact $c_{1+} < 0$ simply follows from the property that $n_\varepsilon < 1$.

3. EXISTENCE

In order to apply the geometric theory of singular perturbations, we first write (2.1)–(2.2) into the following first order system:

$$\begin{cases} \delta v_1' = v_2, \\ \delta v_2' = v_3, \\ \delta v_3' = v_4, \\ \delta^3 v_4' = -\rho v_1 + \delta v_3 + \mu v_4 + \dfrac{\rho n}{1 + \nu(1 - \delta\tau\, \rho_1 v_3)}, \\ n' = m, \\ \delta^3 m' = G(v_1, v_2, v_3, v_4, n, m, \delta), \end{cases} \tag{3.1}$$

where $\delta := \varepsilon^{1/3}$,

$$\begin{aligned} &G(v_1, v_2, v_3, v_4, n, m, \delta) \\ &\quad := \frac{1}{(1 + G_1(v_3, n, \delta))} \bigg\{ -n(1-n) + m - \frac{\delta^3 \alpha\tau\rho_1 m v_4}{1+\gamma n} - \frac{\delta^3 \alpha\gamma m^2 (1 - \tau\rho_1\, \delta v_3)}{(1+\gamma n)^2} \\ &\qquad + \frac{2\alpha\tau\rho_1 \gamma m n\, \delta^3 v_4}{(1+\gamma n)^2} + \frac{2\alpha\gamma^2\, \delta^3 m^2 n (1 - \tau\rho_1\, \delta v_3)}{(1+\gamma n)^3} - \frac{\alpha\tau\rho_1 n}{1+\gamma n} \\ &\qquad \times \bigg[-\rho v_1 + \delta v_3 + \mu v_4 + \frac{\rho n}{1 + \nu(1 - \delta\tau\, \rho_1 v_3)} \bigg] \bigg\}, \end{aligned}$$

$$G_1(v_3, n, \delta) := \frac{\alpha\gamma n (1 - \delta\tau\, \rho_1 v_3)}{(1+\gamma n)^2}.$$

Let $x = (v_4, m)$, $y = (v_1, v_2, v_3)$. The above system becomes

$$\begin{cases} \delta^3 x' = f(x, y, n, \delta), \\ \delta y' = g(x, y, n, \delta), \\ n' = h(x, y, n, \delta), \end{cases} \tag{3.2}$$

where f, g, h are defined by the right-hand sides of (3.1) respectively. We note that z is the slow variable and x and y are fast variables, however, of different scales with respect to small δ, that is, x is faster than y.

One important component of the geometric theory of singular perturbations is the Fenichel's invariant manifold theorem which requires the normal hyperbolicity of the critical manifold in a singularly perturbed system ([6]). Due to the appearance of two time scales, the critical manifold associated to (3.2) fails to be normally hyperbolic. To resolve this problem, we first introduce the new independent variable $z_1 := z/\delta^2$ to reduce (3.2) to

$$\begin{cases} \delta \dfrac{dx}{dz_1} = f(x, y, n, \delta), \\ \dfrac{dy}{dz_1} = \delta g(x, y, n, \delta), \\ \dfrac{dn}{dz_1} = \delta^2 h(x, y, n, \delta), \end{cases} \tag{3.3}$$

in which x becomes the only fast variable and (y, n) is the slow variable. Let $\delta = 0$ in (3.3). The associated critical manifold can be solved from $f(x, y, n, 0) = 0$ which yields

$$x = x_0(y, n) = \left(\frac{\rho}{\mu}\left(v_1 - \frac{n}{1+v}\right), n(1-n)\right).$$

Consider the following bounded portion of this manifold:

$$X_0 = \left\{(x, y, n) : x = x_0(y, n), \left|v_1 - \frac{n}{1+v}\right| \leqslant 1, |v_2| \leqslant 1, |v_3| \leqslant 1, -\eta \leqslant n \leqslant 2\right\},$$

where η is chosen to be a small (but fixed) positive number that satisfies $1 + \gamma n > 1/2$ and $1 + \frac{\gamma n}{(1+\gamma n)^2} > 1/2$ when $n > -\eta$. Such choice of η ensures the smoothness of G in (3.1) in the vicinity of X_0. Since

$$D_x f(x, y, n, 0) = \begin{pmatrix} \mu & 0 \\ * & \dfrac{1}{1 + \frac{\alpha \gamma n}{(1+\gamma n)^2}} \end{pmatrix},$$

X_0 satisfies the normal hyperbolic condition required by the Fenichel's theorem, and hence there exists a normally hyperbolic invariant manifold X_δ of (3.3), called *slow manifold*, such that

$$X_\delta = \left\{(x, y, n) \colon x = x_\delta(y, n), \left|v_1 - \frac{n}{1+v}\right| \leqslant 1, |v_2| \leqslant 1, |v_3| \leqslant 1, -\eta \leqslant n \leqslant 2\right\},$$

where $x_\delta(y, n) := (v_4(y, n, \delta), m(y, n, \delta)) = x_0(y, n) + O(\delta)$. In order to obtain (2.5) we need a more accurate formula for x_δ. By the (local) invariance of flows of (3.3) on X_δ, we have

$$\begin{cases} v_4(y, n, \delta) = \dfrac{\rho}{\mu}\left(v_1 - \dfrac{n}{1+\nu}\right) - \dfrac{1}{\mu}\left[1 + \dfrac{\nu\tau n}{(1+\nu)^2}\right] v_3\delta \\ \qquad + \dfrac{1}{\mu}\left[\dfrac{\rho v_2}{\mu} - \dfrac{\nu^2\tau^2\rho_1 v_3^2 n}{(1+\nu)^3}\right]\delta^2 + O(\delta^3), \\ m(y, n, \delta) = n(1-n) + \dfrac{\alpha\tau v_2 n}{\mu(1+\gamma n)}\delta^2 + O(\delta^3). \end{cases} \tag{3.4}$$

Since the equilibria $(0, 0, 0, 0, 0, 0)$ and $(1/(1+\nu), 0, 0, 0, 1, 0)$ of (3.3) have to lie on the slow manifold X_δ, $v_4(0, 0, 0, 0, \delta) = m(0, 0, 0, 0, \delta) = v_4(\frac{1}{1+\nu}, 0, 0, 1, \delta) = m(\frac{1}{1+\nu}, 0, 0, 1, \delta) = 0$ for all small δ.

Next, we restrict (3.3) to X_δ with $|v_1 - \frac{n}{1+\nu}| \leqslant 1$, $|v_2| \leqslant 1$, $|v_3| \leqslant 1$, $-\eta \leqslant n \leqslant 2$. This yields the slow flow

$$\begin{cases} \dfrac{dy}{dz_1} = \delta g(x_\delta(y, n), y, n, \delta), \\ \dfrac{dn}{dz_1} = \delta^2 h(x_\delta(y, n), y, n, \delta), \end{cases}$$

which, in term of the original independent variable z, reads

$$\begin{cases} \delta \dfrac{dy}{dz} = g(x_\delta(y, n), y, n, \delta), \\ \dfrac{dn}{dz} = h(x_\delta(y, n), y, n, \delta). \end{cases} \tag{3.5}$$

The system (3.5) is again a singularly perturbed system with the fast variable y and the slow variable n, whose critical manifold Y_0 is determined by $g(x_0(y, n), y, n, 0) = 0$. Since

$$g(x_0(y, n), y, n, 0) = \begin{pmatrix} v_2 \\ v_3 \\ \dfrac{\rho}{\mu}\left(v_1 - \dfrac{n}{1+\nu}\right) \end{pmatrix},$$

$$D_y g(x_0(y, n), y, n, 0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{\rho}{\mu} & 0 & 0 \end{pmatrix},$$

we have that

$$Y_0 = \left\{ (y, n) \colon y = y_0(n) = \left(\frac{n}{1+\nu}, 0, 0 \right), -\eta \leqslant n \leqslant 2 \right\},$$

which is also normally hyperbolic. An application of the Fenichel's theorem again yields a slow manifold Y_δ for (3.5) near Y_0 having the form

$$Y_\delta = \left\{ (y, n) \colon y = y_\delta(n) = \left(\frac{n}{1+\nu} + O(\delta), O(\delta), O(\delta) \right), -\eta \leqslant n \leqslant 2 \right\}.$$

Using the local invariance of Y_δ to (3.5) we obtain a more accurate formula for $y_\delta(n) = (v_1(n, \delta), v_2(n, \delta), v_3(n, \delta))$:

$$\begin{cases} v_1(n, \delta) = \dfrac{n}{1+\nu} + O(\delta^3), \\ v_2(n, \delta) = \dfrac{n(1-n)}{1+\nu} \delta + O(\delta^3), \\ v_3(n, \delta) = \dfrac{n(1-n)(1-2n)}{1+\nu} \delta^2 + O(\delta^3). \end{cases} \tag{3.6}$$

Since $(0, 0, 0, 0)$ and $(1/(1+\nu), 0, 0, 1)$ are equilibria of (3.5), it follows that $v_1(0, \delta) = 0$, $v_1(1, \delta) = 1/(1+\nu)$, and $v_i(0, \delta) = v_i(1, \delta) = 0$ $(i = 2, 3)$ for δ sufficiently small.

Now, using (3.4) and (3.6), the restriction of (3.5) on Y_δ reads

$$\frac{dn}{dz} = n(1-n) + N(n, \delta), \tag{3.7}$$

where $N(n, \delta) = O(\delta^3)$ for sufficiently small δ. Since $n = 0$ and $n = 1$ are equilibria of (3.7), $N(0, \delta) = N(1, \delta) = 0$ for all sufficiently small δ. It follows from the smoothness of N with respect to both n and δ that $N(n, \delta) = O(\delta^3)\ |n(1-n)|$ for $-\eta \leqslant n \leqslant 2$ and sufficiently small δ. Hence, for $0 \leqslant n \leqslant 1$, (3.7) can be written as

$$\frac{dn}{dz} = n(1-n)[1 + O(\delta^3)]. \tag{3.8}$$

Now any solution n of (3.8) with $0 < n(0) < 1$ exists on $(-\infty, \infty)$ and satisfies the properties that $0 < n < 1$, $n' > 0$, on $(-\infty, \infty)$, $\lim_{z \to -\infty} n(z) = 0$, and $\lim_{z \to \infty} n(z) = 1$. In particular, the one with $n(0) = 1/2$ is given by

$$n_\delta(z) = \frac{e^{(1+O(\delta^3))\, z}}{1 + e^{(1+O(\delta^3))\, z}}, \qquad -\infty < z < \infty. \tag{3.9}$$

Thus, $(x, y, n) := (x_\delta(y_\delta(n_\delta), n_\delta), y_\delta(n_\delta), n_\delta)$ is a heteroclinic orbit of (3.1) connecting the equilibrium points $(0, 0, 0, 0, 0, 0)$ and $(\frac{1}{1+v}, 0, 0, 0, 1, 0)$ at $z = -\infty$ and $z = \infty$ respectively.

Next, we show that $v_{1,\delta}, v'_{1,\delta} > 0$ on $(-\infty, \infty)$. We note by (3.6) that, on the manifold Y_0,

$$v_1(n, \delta) = \frac{n}{1+v} + V_1(n, \delta),$$

where $V_1(n, \delta) = O(\delta^3)$ is a smooth function of (n, δ). Since $v_1(0, \delta) = 0$ and $v_1(1, \delta) = \frac{1}{1+v}$ for all sufficiently small δ, we have $V_1(0, \delta) = V_1(1, \delta) = 0$ for all sufficiently small δ. Clearly, $V_1(n, 0) = 0$ and therefore $(\partial V_1/\partial n)(n, 0) = 0$ for all $0 \leqslant n \leqslant 1$. It follows that

$$\frac{\partial V_1}{\partial n}(n, \delta) = \int_0^\delta \frac{\partial^2 V_1}{\partial \delta \, \partial n} \, d\delta = O(\delta),$$

from which, we have

$$V_1(n, \delta) = \int_0^n \frac{\partial V_1}{\partial n} \, dn = O(\delta) \, n.$$

Hence

$$\begin{aligned} v_{1,\delta} &= \left(\frac{1}{1+v} + O(\delta) \right) n_\delta, \\ \frac{dv_{1,\delta}}{dz} &= \left(1 + \frac{\partial V_1}{\partial n}(n_\delta, \delta) \right) \frac{dn_\delta}{dz} = (1 + O(\delta)) \frac{dn_\delta}{dz} > 0. \end{aligned} \tag{3.10}$$

Now let $(v_\varepsilon, n_\varepsilon) := (v_{1,\delta}, n_\delta)$. We have by (3.1), (3.4), (3.6), (3.9), (3.10), and $\delta = \varepsilon^{1/3}$ that

$$\begin{cases} v_\varepsilon = \dfrac{n_\varepsilon}{1+v} + O(\varepsilon) = \left(\dfrac{1}{1+v} + O(\varepsilon^{1/3}) \right) n_\varepsilon, \\ \dfrac{dv_\varepsilon}{dz} = \dfrac{n_\varepsilon(1-n_\varepsilon)}{1+v} + O(\varepsilon^{2/3}) = (1 + O(\varepsilon^{1/2})) \dfrac{dn_\varepsilon}{dz}, \\ \dfrac{d^2 v_\varepsilon}{dz^2} = \dfrac{n_\varepsilon(1-n_\varepsilon)(1-2n_\varepsilon)}{1+v} + O(\varepsilon^{1/3}), \\ \dfrac{d^3 v_\varepsilon}{dz^3} = O(1), \\ n_\varepsilon(z) = \dfrac{e^{(1+O(\varepsilon))\,z}}{1 + e^{(1+O(\varepsilon))\,z}}, \\ \dfrac{dn_\varepsilon}{dz} = n_\varepsilon(1-n_\varepsilon)[1 + O(\varepsilon)]. \end{cases} \tag{3.11}$$

Therefore, $(v_\varepsilon, n_\varepsilon)$ is a solution of (2.1)–(2.3) satisfying (2.5) and that $n_\varepsilon(0)=\frac{1}{2}$, $n'_\varepsilon>0$, $v_\varepsilon>0$, $v'_\varepsilon>0$ on $(-\infty,\infty)$. The uniqueness of such a solution follows from the local uniqueness of slow manifolds X_δ and Y_δ and the uniqueness of n_δ. A similar argument used in [1] shows the inequalities in (2.4). This completes the proof of part (i) of Theorem 2.1.

4. GLOBAL UNIQUENESS

In this section we present a global uniqueness result for (2.1)–(2.3) which is more general than that stated in (ii) of Theorem 2.1. A condition for such uniqueness is the following *a prior* uniform boundedness on v with respect to ε.

Assumption A. There is a constant $M>0$ such that if ε is sufficiently small and (v,n) is a solution of (2.1)–(2.3) with $0<n<1$ on $(-\infty,\infty)$, then

$$|v(z)|+\varepsilon^{1/3}\,|v'(z)|+\varepsilon^{2/3}\,|v''(z)|+\varepsilon\,|v'''(z)|\leqslant M, \qquad -\infty<z<\infty.$$

Remark 4.1. Under the Assumption A, we have

$$|n'(z)|\leqslant M', \qquad -\infty<z<\infty,$$

for some constant M' which depends only on M. To see this, we note that $n'(\pm\infty)=0$ implies that $|n'|$ reaches its maximum at some point in $(-\infty,\infty)$ where $n''=m'=0$. It follows from (3.1) that $G=0$ at the maximum point, from which $n'=m$ can be solved as ε is sufficiently small. The desired assertion now follows from Assumption A.

Remark 4.2. The Assumption A is satisfied when $v=0$. This follows from Lemma 5.1 in the Appendix and the fact that the right-hand side of (2.1) is bounded by ρ which is independent of any particular solution (v,n) of (2.1)–(2.3) with $0<n<1$. This fact together with Theorem 4.1 below shows part (ii) of Theorem 2.1.

Theorem 4.1. *Assume the Assumption A. If ε is sufficiently small, then the solution to (2.1)–(2.3) is globally unique in the sense that if (v,n) and $(\tilde{v},\tilde{n})$ are two solutions with $0<\tilde{n}<1$, $0<n<1$ and $\tilde{n}(0)=n(0)$, then $(\tilde{v},\tilde{n})=(v,n)$.*

Proof. Let $(v,n):=(v_\varepsilon,n_\varepsilon)$ be a solution to (2.1)–(2.3). Then (v_1,v_2,v_3,v_4,n,m) with $v_1:=v$ satisfies (3.1). Again let $x=(v_4,m)$ and $y=(v_1,v_2,v_3)$. Then (x,y,n) satisfies (3.2) and (3.3).

Let X_0 be the slow manifold defined in Section 3. We claim that for any given small neighborhood U of X_0, if δ is sufficiently small, then (x, y, n) lies in U.

By introducing the new independent variable $\xi_1 = z_1/\delta$, (3.3) becomes

$$\begin{cases} \dot{v}_1 = \delta^2 v_2, \\ \dot{v}_2 = \delta^2 v_3, \\ \dot{v}_3 = \delta^2 v_4, \\ \dot{n} = \delta^3 m, \\ \dot{v} = -\rho v_1 + \delta v_3 + \mu v_4 + \dfrac{\rho n}{1+\nu(1-\delta\tau\rho_1 v_3)}, \\ \dot{m} = G(v_1, v_2, v_3, v_4, n, m, \delta), \end{cases} \tag{4.1}$$

where $\cdot = d/d\xi_1$. Let

$$\begin{cases} K_1 = -\rho v_1 + \mu v_4 + \dfrac{\rho n}{1+\nu}, \\ K_2 = m - n(1-n). \end{cases}$$

Then the critical manifold X_0 is given by $K_1 = K_2 = 0$. Thus, in order to show the above claim, it suffices to show that for any given small neighborhood V of $(0, 0)$, (K_1, K_2) lies in V provided that δ is sufficiently small.

By Assumption A and Remark 4.1 we have $|x| + |y| + |n| \leqslant M + M'$ for sufficiently small δ. Hence, $|K_1| + |K_2| \leqslant M_1$ for sufficiently small δ, where $M_1 > 0$ is a constant independent of (x, y, z) and δ. It follows from (4.1) that (K_1, K_2) satisfies the equations

$$\begin{cases} \dot{K}_1 = \mu K_1 + O(\delta), \\ \dot{K}_2 = \dfrac{1}{1+\frac{\alpha\gamma n}{(1+\gamma n)^2}} \left\{ -\dfrac{\alpha\tau\rho_1 n}{1+\gamma n} K_1 + K_2 \right\} + O(\delta), \end{cases} \tag{4.2}$$

where $|O(\delta)| \leqslant M_2\delta$, and $M_2 > 0$ is a constant independent of (K_1, K_2) and δ. By writing out the explicit solution form of (4.2), we see easily that if δ is sufficiently small and (K_1, K_2) were not in the interior of V, then there would exist a time τ_1 at which $|K_1(\tau_1)| + |K_2(\tau_1)| = 2M_1$. This leads to a contradiction, thereby proving the claim.

It follows from the above claim and the Fenichel's theorem that the connecting orbit (x, y, z) has to lie on X_δ for sufficiently small δ, and therefore, its (y, n) components satisfy (3.5) on $(-\infty, \infty)$.

Similarly, by introducing the new independent variable $\xi = z/\delta$, (3.5) becomes

$$\begin{cases} \dfrac{dy}{d\xi} = g(x_\delta(y, n), y, n, \delta), \\ \dfrac{dn}{d\xi} = \delta h(x_\delta(y, n), y, n, \delta). \end{cases} \tag{4.3}$$

Let $K_3 = v_1 - \frac{n}{1+v}$. Clearly, $|K_3| \leqslant M_3$ for some constant $M_3 > 0$. From (4.3) we have

$$\begin{cases} \dfrac{dK_3}{d\xi} = v_2 + O(\delta), \\ \dfrac{dv_2}{d\xi} = v_3, \\ \dfrac{dv_3}{d\xi} = \dfrac{\rho}{\mu} K_3 + O(\delta). \end{cases} \tag{4.4}$$

Note that the slow manifold Y_0 is given by $K_3 = v_2 = v_3 = 0$. A similar argument as above shows that (K_3, v_2, v_3) has to lie in a small neighborhood of $\{K_3 = v_2 = v_3 = 0\}$ for all $\xi \in (-\infty, \infty)$. Namely, (y, n) has to lie in a small neighborhood of Y_0 and hence lie on Y_δ. This shows that the component n satisfies (3.8) on $(-\infty, \infty)$.

We note that all constants M_i $(i = 1, 2, 3)$ involved in the above arguments are independent of any particular connecting orbits (v, n) to (2.1)–(2.3) with $0 < n < 1$. Therefore, if (v, n) and $(\tilde{v}, \tilde{n})$ are two different such solutions with $n(0) = \tilde{n}(0) = 1/2$, then their corresponding system counterparts (x, y, n) and $(\tilde{x}, \tilde{y}, \tilde{n})$ lie on X_δ, (y, n) and $(\tilde{y}, \tilde{n})$ lie on Y_δ, with both n and $\tilde{n}$ satisfying (3.8) which is a first order autonomous scalar equation of n. Since $n \equiv \tilde{n}$, we have $y \equiv \tilde{y}$ and $x \equiv \tilde{x}$ by the local uniqueness of X_δ and Y_δ. Hence, $v \equiv \tilde{v}$. This completes the proof of part (ii) of Theorem 2.1. $\square$

5. APPENDIX

Lemma 5.1. *There exists a constant $M > 0$ such that if ε is sufficiently small, then for any $f \in C(-\infty, \infty)$ with $\sup_{z \in (-\infty, \infty)} |f(z)| < \infty$ the equation*

$$\beta \varepsilon^2 \frac{d^4 v}{dz^4} - \mu \varepsilon \frac{d^3 v}{dz^3} - \varepsilon \frac{d^2 v}{dz^2} + \rho v = f(z) \tag{5.1}$$

has a unique bounded solution v on $(-\infty, \infty)$, which satisfies

$$|v| + \varepsilon^{1/3}\,|v'| + \varepsilon^{2/3}\,|v''| + \varepsilon\,|v'''| \leqslant M \max_{z\in(-\infty,\infty)} |f(z)|. \tag{5.2}$$

Furthermore, if $\lim_{z\to\infty} f(z) = f_\infty$ *exists, then*

$$\lim_{z\to\infty} (v, v', v'', v''')(z) = (f_\infty/\rho, 0, 0, 0). \tag{5.3}$$

We first proof the following lemma.

Lemma 5.2. *For ε sufficiently small, the equation $p(\lambda) := \beta\varepsilon^2\lambda^4 - \mu\varepsilon\lambda^3 - \varepsilon\lambda^2 + \rho = 0$ admits two complex conjugate eigenvalues $\lambda_1 = \bar{\lambda}_2$ and $\lambda_2 = a + ib$, and two real eigenvalues $0 < \lambda_3 < \lambda_4$, satisfying*

$$\begin{cases} a = -\dfrac{1}{2}\sqrt[3]{\dfrac{\rho}{\mu\varepsilon}}\,(1+O(\varepsilon^{1/3})), \\ b = \dfrac{\sqrt{3}}{2}\sqrt[3]{\dfrac{\rho}{\mu\varepsilon}}\,(1+O(\varepsilon^{1/3})), \\ \lambda_3 = \sqrt[3]{\dfrac{\rho}{\mu\varepsilon}}\,(1+O(\varepsilon^{1/3})), \\ \lambda_4 = \dfrac{\mu}{\beta\varepsilon}(1+O(\varepsilon)), \end{cases} \qquad \textit{as } \varepsilon \to 0. \tag{5.4}$$

Proof. Let ε be sufficiently small. Then, $p(\lambda) = \varepsilon\lambda^3[\beta\varepsilon\lambda - \mu + O(\varepsilon)]$ on the interval $[\frac{\mu}{2\beta\varepsilon}, \frac{2\mu}{\beta\varepsilon}]$ and $p(\lambda) = -\mu\varepsilon[\lambda^3 - \frac{\rho}{\mu\varepsilon}(1+O(\varepsilon^{1/3}))]$ on $[\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}, 2\sqrt[3]{\frac{\rho}{\mu\varepsilon}}]$ respectively. Hence by the intermediate value theorem, λ_4 and λ_3 exist in the intervals $[\frac{\mu}{2\beta\varepsilon}, \frac{2\mu}{\beta\varepsilon}]$ and $[\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}, 2\sqrt[3]{\frac{\rho}{\mu\varepsilon}}]$ respectively, and satisfy the asymptotic formulas in (5.4). Similarly, for λ in the region $|\lambda - (-\frac{1}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}} + i\frac{\sqrt{3}}{2}\sqrt[3]{\frac{\rho}{\mu\varepsilon}})| \leqslant \frac{1}{5}\sqrt[3]{\frac{\rho}{\mu\varepsilon}}$ on the complex plane, $p(\lambda) = -\mu\varepsilon[\lambda^3 - \frac{\rho}{\mu\varepsilon}(1+O(\varepsilon^{1/3}))]$. Hence, by the Rouche's theorem, λ_2 exists in this region and satisfies the asymptotic formula in (5.4). The proof of the lemma is now completed by letting $\lambda_1 = \bar{\lambda}_2$. □

Proof of Lemma 5.1. We first write (5.1) as the equivalent system

$$\phi' = A\phi + F(z), \tag{5.5}$$

where $\phi := (v, v', v'', v''')^\top$, A is the corresponding 4×4 coefficient matrix whose entries are independent of z, and $F(z) = (0, 0, 0, \frac{f(z)}{\beta\varepsilon^2})^\top$. The eigenvalues of A are λ_i $(i = 1, 2, 3, 4)$ as described in Lemma 5.2 with the associated eigenvectors $(1, \lambda_i, \lambda_i^2, \lambda_i^3)^\top$. Therefore,

$$T^{-1}AT = \Lambda := \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \qquad \text{where}$$

$$T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ a & b & \lambda_3 & \lambda_4 \\ a^2 - b^2 & 2ab & \lambda_3^2 & \lambda_4^2 \\ a^3 - 3ab^2 & 3a^2b - b^3 & \lambda_3^3 & \lambda_4^3 \end{pmatrix}.$$

We note that the first two columns of T are the real and the imaginary parts of the complex eigenvector $(1, \lambda_2, \lambda_2^2, \lambda_2^3)^\top$ respectively.

Let $\phi = Tx$ and $x = (x_1, x_2, x_3, x_4)^\top$. Then (5.5) becomes

$$\begin{cases} x_1' = ax_1 + bx_2 + \dfrac{\alpha_1}{\beta\varepsilon^2} f(z), \\ x_2' = -bx_1 + ax_2 + \dfrac{\alpha_2}{\beta\varepsilon^2} f(z), \\ x_3' = \lambda_3 x_3 + \dfrac{\alpha_3}{\beta\varepsilon^2} f(z), \\ x_4' = \lambda_4 x_4 + \dfrac{\alpha_4}{\beta\varepsilon^2} f(z), \end{cases} \tag{5.6}$$

where $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\top$ is the last column of T^{-1} given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} := T^{-1}e_4 = \begin{pmatrix} \dfrac{\lambda_4 + \lambda_3 - 2a}{[(\lambda_4 - a)^2 + b^2][(\lambda_3 - a)^2 + b^2]} \\ \dfrac{(\lambda_3 - a)\,\lambda_4 - a\lambda_3 + a^2 - b^2}{b[(\lambda_4 - a)^2 + b^2][(\lambda_3 - a)^2 + b^2]} \\ \dfrac{1}{(\lambda_3 - \lambda_4)[(\lambda_3 - a)^2 + b^2]} \\ \dfrac{1}{(\lambda_4 - \lambda_3)[(\lambda_4 - a)^2 + b^2]} \end{pmatrix}, \qquad \text{with} \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.7}$$

It follows that (5.6) admits a unique bounded solution over $(-\infty, \infty)$ given by

$$x_1(z) = \frac{1}{\beta\varepsilon^2} \int_{-\infty}^{z} a^{a(z-s)} [\alpha_1 \cos b(z-s) + \alpha_2 \sin b(z-s)] \, f(s) \, ds, \tag{5.8}$$

$$x_2(z) = \frac{1}{\beta\varepsilon^2} \int_{-\infty}^{z} e^{a(z-s)} [-\alpha_1 \sin b(z-s) + \alpha_2 \cos b(z-s)] \, f(s) \, ds, \tag{5.9}$$

$$x_3(z) = -\frac{\alpha_3}{\beta\varepsilon^2} \int_{z}^{\infty} e^{\lambda_3(z-s)} f(s) \, ds, \tag{5.10}$$

$$x_4(z) = -\frac{\alpha_4}{\beta\varepsilon^2} \int_{z}^{\infty} e^{\lambda_4(z-s)} f(s) \, ds. \tag{5.11}$$

Furthermore, if $\lim_{z \to \infty} f(z) = f_\infty$ exists, then

$$x(z) \to -\frac{f_\infty}{\beta\varepsilon^2} \Lambda^{-1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\top \qquad \text{as} \quad z \to \infty. \tag{5.12}$$

From (5.4) and (5.7) it follows that, as $\varepsilon \to 0$,

$$\alpha_1 \sim \frac{1}{\lambda_4[(\lambda_3 - a)^2 + b^2]}, \qquad \alpha_2 \sim \frac{\lambda_3 - a}{b\lambda_4[(\lambda_3 - a)^2 + b^2]},$$
$$\alpha_3 \sim \frac{-1}{\lambda_4[(\lambda_3 - a)^2 + b^2]}, \qquad \alpha_4 \sim \frac{1}{\lambda_4^3},$$

and hence there is a constant $M > 0$ independent of ε such that if ε is sufficiently small, then

$$|\alpha_1| + |\alpha_2| + |\alpha_3| \leqslant M\varepsilon^{\frac{5}{3}}, \qquad |\alpha_4| \leqslant M\varepsilon^3.$$

This together with (5.8)–(5.11) yields that, on $(-\infty, \infty)$,

$$|x_1| + |x_2| + |x_3| \leqslant M \, |f|_0, \qquad |x_4| \leqslant M\varepsilon^2 \, |f|_0, \tag{5.13}$$

where $|f|_0 = \sup_{z \in (-\infty, \infty)} |f(z)|$ and the constant M might be changed but still independent of ε and f.

Using the transformation between v and x, (5.4) and (5.13), we have, on $(-\infty, \infty)$,

$$\begin{aligned}
|v| &\leqslant |x_1| + |x_3| + |x_4| \leqslant M \, |f|_0, \\
|v'| &\leqslant M(\varepsilon^{-\frac{1}{3}} \, |x_1| + \varepsilon^{-\frac{1}{3}} \, |x_2| + \varepsilon^{-\frac{1}{3}} \, |x_3| + \varepsilon^{-1} \, |x_4|) \leqslant M\varepsilon^{-\frac{1}{3}} \, |f|_0, \\
|v''| &\leqslant M(\varepsilon^{-\frac{2}{3}} \, |x_1| + \varepsilon^{-\frac{2}{3}} \, |x_2| + \varepsilon^{-\frac{2}{3}} \, |x_3| + \varepsilon^{-2} \, |x_4|) \leqslant M\varepsilon^{-\frac{2}{3}} \, |f|_0, \\
|v'''| &\leqslant M(\varepsilon^{-1} \, |x_1| + \varepsilon^{-1} \, |x_2| + \varepsilon^{-1} \, |x_3| + \varepsilon^{-3} \, |x_4|) \leqslant M\varepsilon^{-1} \, |f|_0,
\end{aligned}$$

that is, (5.2) holds. Finally, (5.3) follows from (5.12) and the limit that

$$\phi(z) = Tx(z) \to -\frac{f_\infty}{\beta\varepsilon^2} T\,\Lambda^{-1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\top = -\frac{f_\infty}{\beta\varepsilon^2}(\Lambda^{-1}T)(T^{-1}e_4)$$

$$= -\frac{f_\infty}{\beta\varepsilon^2}\Lambda^{-1}e_4 = -\frac{f_\infty}{\beta\varepsilon^2}\left(-\frac{\beta\varepsilon^2}{\rho}, 0, 0, 0\right)^\top = \left(\frac{f_\infty}{\rho}, 0, 0, 0\right)^\top,$$

as $z \to \infty$. This completes the proof of Lemma 5.1. □

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