COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **3**, Number **2**, June **2004**

Website: http://AIMsciences.org

pp. **175–182**

MULTIPLE POSITIVE PERIODIC SOLUTIONS FOR A DELAY HOST MACROPARASITE MODEL

Shangbing Ai

Department of Mathematical Sciences
University of Alabama in Huntsville, Huntsville, AL 35899

(Communicated by Carmen Chicone)

Abstract. A scalar non-autonomous periodic differential equation with delays arising from a delay host macroparasite model is studied. Two results are presented for the equation to have at least two positive periodic solutions: the hypotheses of the first result involve delays, while the second result holds for arbitrary delays.

1. **Introduction.** In recent years, periodic population dynamics with delays has become a very popular subject, and various models have been studied (See [1, 6, 8, 9, 10, 12, 13] and the references therein). One of the important mathematical problems for such models is to show the existence of positive periodic solutions. For example, the scalar equation with a delay r

$$x'(t) = a(t)x(t)\Big[\frac{1}{(1+\kappa x(t-r))^n} - \frac{1}{(1+\kappa_1 x(t-r))^n} - c(t)\Big] \tag{1.1}$$

arises from a delay host macroparasite model (See [10, 12] and the references therein), where $\kappa < \kappa_1$ and r are positive numbers, n is a positive integer, a and c are continuous and positive ω-periodic functions on $(-\infty, \infty)$, c is not identically constant, and x is the number of sexually mature worms in the human community of some fixed size. Let

$$f(x) = \frac{1}{(1+\kappa x)^n} - \frac{1}{(1+\kappa_1 x)^n}, \qquad \text{and} \qquad x^* = \frac{\kappa_1^{\frac{1}{n+1}} - \kappa^{\frac{1}{n+1}}}{\kappa_1 \kappa^{\frac{1}{n+1}} - \kappa\kappa_1^{\frac{1}{n+1}}}.$$

We see that $f > 0$ on $(0, \infty)$, $f(0) = \lim_{x\to\infty} f(x) = 0$ and f has a unique maximum at x^*. Lemma 2.3 shows that if $\max_{0\le t\le\omega} c(t) < f(x^*)$ and $r = 0$, then (1.1) has two (and only two) positive ω-periodic solutions, one lying in $(0, x^*)$ and the other lying in (x^*, ∞). The goal of the paper is to show that, under some further conditions, (1.1) has (at least) two positive ω-periodic solutions for $r \neq 0$. Roughly speaking, we present two such results: Theorems 2.1 and 3.1. Theorem 2.1 holds for "small" r, while Theorem 3.1 holds for all r.

In fact, we study a more general equation:

$$x'(t) = a(t)g(t, x(t - r_0(t)))[f(x(t - r(t))) - c(t)], \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34K13, 92B05, 92D25.
Key words and phrases. Delay host macroparasite model, multiple positive periodic solutions, Schauder's and Krasnosel'skii's fixed point theorems.

where a and c are the same as in (1.1), r_0 and r are continuous ω-periodic functions on $(-\infty, \infty)$, g is positive and continuous on $(-\infty, \infty) \times (0, \gamma)$ and ω-periodic with respect to t, f is continuous on $(0, \infty)$ and increasing on $(0, x^*)$ and decreasing on (x^*, γ) for some $x^* \in (0, \gamma)$; and $\lim_{x \to 0^+} f(x) = \lim_{x \to \gamma^-} f(x) = 0$, where γ is a positive number that could be infinity. In Section 2 we study (1.2) with r not too large. We regard (1.2) as a perturbation to the $r = 0$ case and use a result, Lemma 2.3, for an ode and Schauder's fixed point theorem to show Theorem 2.1 whose hypotheses involve r. In Section 3, we restrict ourself, for simplicity, to the case $g(t, x) = x$, $r_0 \equiv 0$ in (1.2) and obtain a simple sufficient condition in Theorem 3.1 that is independent of r. The proof of this result, which is motivated from that of [6], is based on Krasnosel'skii's fixed point theorem on a cone in [7]. We point out in Remark 3.2 that these two results are independent of each other. Some extensions of these results are given. A short summary is given in Section 4, and the proofs that the operators used in the proofs of Theorems 2.1 and 3.3 are continuous and compact are given in the appendix.

In the rest of the paper, we use C_ω to denote the Banach space of continuous ω-periodic function on $(-\infty, \infty)$ with the supremum norm $\|\cdot\|_0$. Let $0 < d_1 < \bar{d}_1 < x^* < d_2 < \bar{d}_2 < \gamma$ be real numbers such that $f(d_1) = f(\bar{d}_2) = c_1 := \min_{0 \le t \le \omega} c(t)$, and $f(\bar{d}_1) = f(d_2) = \|c\|_0$ whenever they are defined. We remark that a necessary condition for (1.2) to have a positive solution x in C_ω is $c_1 < f(x^*)$ and any such a solution x satisfies $\|x\|_0 \ge d_1$ and $\min_{0 \le t \le \omega} x(t) \le \bar{d}_2$.

2. **The r "Small" Case.** We start this section with the main result and a corollary.

THEOREM 2.1. *Assume that $\|c\|_0 < f(x^*)$ and f is differentiable on $(0, \gamma)$. Given a number $0 < \eta < \min\{c_1, f(x^*) - \|c\|_0\}$, let $0 < D_1 < \bar{D}_1 < x^* < D_2 < \bar{D}_2 < \gamma$ be real numbers such that $f(D_1) = f(\bar{D}_2) = c_1 - \eta$ and $f(\bar{D}_1) = f(D_2) = \|c\|_0 + \eta$.*

(i) Let $M_0 = \max\{g(t, x) : t \in [0, \omega], x \in [D_1, \bar{D}_1]\}$, $M_1 = M_0(\|c\|_0 - c_1 + 2\eta)$, and $M_2 = \max\{f'(x) : x \in [D_1, \bar{D}_1]\}$. If $\max_{0 \le t \le \omega} \left| \int_{t-r(t)}^{t} a(s)\, ds \right| \le \eta / M_1 M_2$, then (1.2) has a solution x_1 in C_ω satisfying $D_1 \le x_1 \le \bar{D}_1$ on $(-\infty, \infty)$.

(ii) Let $M_3 = \max\{g(t, x) : t \in [0, \omega], x \in [D_2, \bar{D}_2]\}$, $M_4 = M_3(\|c\|_0 - c_1 + 2\eta)$, and $M_5 = \max\{|f'(x)| : x \in [D_2, \bar{D}_2]\}$. If $\max_{0 \le t \le \omega} \left| \int_{t-r(t)}^{t} a(s)\, ds \right| \le \eta / M_4 M_5$, then (1.2) has a solution x_2 in C_ω satisfying $D_2 \le x_2 \le \bar{D}_2$ on $(-\infty, \infty)$.

COROLLARY 2.2. *Use the same notations as in Theorem 2.1. If $\|c\|_0 < f(x^*)$ and $\|a\|_0 \cdot \|r\|_0 \le \eta / \max\{M_1 M_2, M_4 M_5\}$, then (1.2) (with any $r_0 \in C_\omega$) has at least two solutions x_1 and x_2 in C_ω satisfying $D_1 \le x_1 \le \bar{D}_1$ and $D_2 \le x_2 \le \bar{D}_2$ on $(-\infty, \infty)$. In particular, if $\|c\|_0 < f(x^*)$ and $\omega \|a\|_0 \le \eta / \max\{M_1 M_2, M_4 M_5\}$, then the same conclusion holds for (1.2) with any $r_0 \in C_\omega$ and r replaced by any constant.*

Proof. The first part of the corollary directly follows from Theorem 2.1. In the second part, assume that $r \equiv \bar{r}$ for some real number $\bar{r}$. If $|\bar{r}| \le \omega$, then the required conclusion follows directly from that in the first part of the corollary. If $|\bar{r}| > \omega$, we write $\bar{r} = k\omega + \tilde{r}$ where k is an integer and $|\tilde{r}| \le \omega$, and then we have that the equation (1.2) with $r(t)$ replaced by $\tilde{r}$ has two solutions in C_ω, and it is easy to check that these solutions are also solutions to (1.2) with $r(t)$ replaced by $\bar{r}$. This shows the corollary. $\square$

We need the following lemma to show Theorem 2.1.

LEMMA 2.3. *Let f, c, η, D_1, $\bar{D}_1$, D_2 and $\bar{D}_2$ be the same as in Theorem 2.1. Then, the ode*

$$y'(t) = A(t)B(y(t))[f(y(t)) - (c(t) + \delta(t))], \tag{2.1}$$

where B is continuous and positive on $(0, \infty)$, A and δ are in C_ω, A is positive, and $\|\delta\|_0 \le \eta$, has two and only two positive solutions y_1 and y_2 in C_ω; furthermore, $D_1 \le y_1 \le \bar{D}_1$ and $D_2 \le y_2 \le \bar{D}_2$ on $(-\infty, \infty)$.

Proof. Observe that if y is a solution of (2.1), then $y'(t) < 0$ whenever $0 < y(t) < D_1$ or $y(t) > \bar{D}_2$, and $y'(t) > 0$ when $\bar{D}_1 < y(t) < D_2$. It follows that if y is a positive solution of (2.1) in C_ω, then either $D_1 \le y \le \bar{D}_1$ or $D_2 \le y \le \bar{D}_2$ on $(-\infty, \infty)$. Therefore, it suffices to show that (2.1) has only two solutions y_1 and y_2 such that $D_1 \le y_1 \le \bar{D}_1$ and $D_2 \le y_2 \le \bar{D}_2$ on $(-\infty, \infty)$.

We first show the existence of y_1. Let y_α denote the solution of (2.1) with $y(0) = \alpha$ and $h(\alpha) = y_\alpha(\omega) - \alpha$. Then, for sufficiently small $\epsilon > 0$, $y_{D_1-\epsilon}(t) < D_1 - \epsilon$ and $y_{\bar{D}_1+\epsilon}(t) > \bar{D}_1 + \epsilon$ for all $t > 0$, and so $h(D_1 - \epsilon) < 0$ and $h(\bar{D}_1 + \epsilon) > 0$. Since h is continuous on $[D_1 - \epsilon, \bar{D}_1 + \epsilon]$, the intermediate value theorem yields that there exists $\tilde{\alpha} \in (D_1 - \epsilon, \bar{D}_1 + \epsilon)$ such that $h(\tilde{\alpha}) = 0$ and hence $y_{\tilde{\alpha}}(\omega) = \tilde{\alpha}$. Therefore, $y_{\tilde{\alpha}}$ is in C_ω, and $D_1 \le y_{\tilde{\alpha}} \le \bar{D}_1$ on $(-\infty, \infty)$. We now show that $y_1 := y_{\tilde{\alpha}}$ is the unique solution of (2.1) lying in $[D_1, \bar{D}_1]$ on $(-\infty, \infty)$. Assume that it is false and let $\bar{y}$ be another such a solution of (2.1). Without loss of generality we assume that $y_1(t) > \bar{y}(t)$ for all t. Then, for all t,

$$\frac{d}{dt}\Big[L(y_1(t)) - L(\bar{y}(t))\Big] = A(t)[f(y_1(t)) - f(\bar{y}(t))] > 0,$$

where $L(y) := \int_{d_1}^{y} (1/B(u))\,du$, and so $L(y_1) - L(\bar{y})$ is increasing on $(-\infty, \infty)$, contradicting that $L(y_1) - L(\bar{y})$ is ω-periodic. This shows the uniqueness of y_1.

In a similar manner we show that (2.1) has a unique solution $y_2 \in C_\omega$ such that $D_2 \le y_2 \le \bar{D}_2$ on $(-\infty, \infty)$. This completes the proof of Lemma 2.3. □

Proof of Theorem 2.1. Since the proofs for (i) and (ii) are similar, we only show (i). Let $\mathcal{S}$ be a convex and closed subset of C_ω consisting of $x \in C_\omega$ such that $D_1 \le x(t) \le \bar{D}_1$ and $|x(t) - x(t - r(t))| \le \eta/M_2$ for $t \in (-\infty, \infty)$. For each given $x \in \mathcal{S}$, we consider the ode

$$y'(t) = a(t)g(t, x(t - r_0(t)))\Big[f(y(t)) - \Big(c(t) + f(x(t)) - f(x(t - r(t)))\Big)\Big]. \tag{2.2}$$

By the mean value theorem, we have $f(x(t)) - f(x(t - r(t))) = f'(\xi)(x(t) - x(t - r(t)))$ for some $\xi \in (D_1, \bar{D}_1)$, and so $|f(x(t)) - f(x(t - r(t)))| \le M_2|x(t) - x(t - r(t))| \le \eta$, and so

$$c_1 - \eta \le c(t) + f(x(t)) - f(x(t - r(t))) \le \|c\|_0 + \eta. \tag{2.3}$$

Therefore, an application of Lemma 2.3 with $A(t) = a(t)g(t, x(t - r_0(t)))$, $B \equiv 1$ and $\delta(t) = f(x(t)) - f(x(t - r(t)))$ yields that (2.2) has a unique solution $y \in C_\omega$ satisfying $D_1 \le y \le \bar{D}_1$. From (2.2) and (2.3) it follows that

$$|y(t) - y(t - r(t))| = \Big|\int_{t-r(t)}^{t} y'(s)\,ds\Big| \le M_0(\|c\|_0 - c_1 + 2\eta)\Big|\int_{t-r(t)}^{t} a(s)\,ds\Big| \le \eta/M_2.$$

(The first inequality above follows because both $f(y(t))$ and $c(t) + f(x(t)) - f(x(t - r(t))$ lie in $[c_1 - \eta, \|c\|_0 + \eta]$.) Hence, we have $y \in \mathcal{S}$.

We now define a mapping T from $\mathcal{S}$ into $\mathcal{S}$ by $Tx = y$, where y is the solution of (2.2) obtained in the above paragraph. Since, for a given $x \in \mathcal{S}$, such y is unique, T

is well defined on $\mathcal{S}$. Note that $(Tx)'$ is uniformly bounded for $x \in \mathcal{S}$. Then, an easy exercise shows that T is continuous and compact on $\mathcal{S}$ (see Appendix). Therefore, by Schauder's fixed point theorem T has at least one fixed point in $\mathcal{S}$. Let x_1 be such a fixed point, which gives a solution of (1.2) as described in (i) of Theorem 2.1, thereby completing the proof of Theorem 2.1.

Corollary 2.4. *In addition to the assumptions in Theorem 2.1, assume further that c is a constant function. Then $x_1 \equiv x_{10}$ and $x_2 \equiv x_{20}$ where x_{10} and x_{20} are the roots of $f(x) - c = 0$ lying in $(0, x^*)$ and (x^*, γ) respectively.*

Proof. In the proof of Theorem 2.1, we have in this special situation $Tx \equiv c$ for any $x \in \mathcal{S}$ due to the uniqueness of the solution of (2.2) in $\mathcal{S}$ for each given $x \in \mathcal{S}$. This shows that $x_1 \equiv x_{10}$. A similar argument gives $x_2 \equiv x_{20}$. □

With a slight modification of its proof, Theorem 2.1 can be generalized to the equation

$$\begin{aligned} x'(t) =& a(t) g(t, x(t - r_0(t, x(t))) \\ & \cdot \Big[F\Big(x(t - r_1(t, x(t), x'(t))), \cdots, x(t - r_m(t, x(t), x'(t))) \Big) - c(t) \Big], \end{aligned} \tag{2.4}$$

where r_i $(i = 0, 1, \cdots, m)$ are continuous and $r_i(t+\omega, x, y) = r_i(t, x, y)$, and $f(x) := F(x, \cdots, x)$ for $x \in (0, \gamma)$ allows to have more oscillations in $(0, \gamma)$. In particular, the following theorem can be proved similarly:

Theorem 2.5. *Let $m = 1$ and $f(x) := F(x)$ in (2.4). Assume that f has n local maxima on $(0, \gamma)$ at $\beta_1 < \cdots < \beta_{2n-1}$ and $n-1$ local minima at $\beta_2 < \cdots < \beta_{2(n-1)}$ satisfying $\beta_{2i-1} < \beta_{2i}$ $(i = 1, \cdots, n-1)$. Let $\beta_0 = 0$ and $\beta_{2n} = \gamma$. Assume that $\|c\|_0 < \min\{f(\beta_{2i-1}) : i = 1, \cdots, n\}$. Let η be a positive number such that $\eta < c_1$ and $\eta < \min\{f(\beta_{2i-1}) : i = 1, \cdots, n\} - \|c\|_0$. For $i = 1, \cdots, 2n$, let $\beta_{i-1} < D_i < \bar{D}_i < \beta_i$ such that $f(D_i) = c_1 - \eta$ and $f(\bar{D}_i) = \|c\|_0 + \eta$ if i is odd, and $f(\bar{D}_i) = \|c\|_0 + \eta$ and $f(D_i) = c_1 - \eta$ if i is even. Let*

$$\begin{aligned} M_0 &= \max \Big\{ g(t, x) : t \in [0, \omega], x \in \cup_{i=1}^{2n} [D_i, \bar{D}_i] \Big\}, \\ M_1 &= M_0 (\|c\|_0 - c_1 + 2\eta), \\ M_2 &= \max \Big\{ |f'(x)| : x \in \cup_{i=1}^{2n} [D_i, \bar{D}_i] \Big\}, \\ \bar{r} &= \max \Big\{ |r_1(t, x, y)| : t \in [0, \omega], x \in \cup_{i=1}^{2n} [D_i, \bar{D}_i], |y| \le \|a\|_0 M_1 \Big\}. \end{aligned}$$

If $\bar{r}\|a\|_0 \le \eta / M_1 M_2$, then (2.4) has at least $2n$ solutions $x_1, \cdots, x_{2n}$ in C_ω such that $D_i \le x_i \le \bar{D}_i$ on $(-\infty, \infty)$ for $i = 1, 2, \cdots, 2n$.

3. **The r Arbitrary Case.** For simplicity of the statement of the main result in this section, we consider the equation (1.2) with $g(t, x) = x$ and $r_0 \equiv 0$. The main result is as follows.

Theorem 3.1. *Let $g(t, x) = x$, $r_0 \equiv 0$ and $\gamma = \infty$. Let $\bar{a} = \int_0^\omega a(s)\, ds$ and $\bar{b} = \int_0^\omega b(s)\, ds$ where $b(t) = a(t) c(t)$. Let $\sigma = e^{-\bar{b}}$ and $p = \frac{\sigma}{1-\sigma}$.*

If $\sigma p f(\sigma x^*) \bar{a} \ge 1$, then (1.2) has at least two solutions x_1 and x_2 in C_ω satisfying $0 < x_1 < x^*$, $\sigma x^* < x_2 < \infty$ on $(-\infty, \infty)$, $\|x_1\|_0 \ge d_1$ and $\|x_2\|_0 > x^*$.

Remark 3.2. We note that Theorem 3.1 does not imply Theorem 2.1. To see this, let us consider the case where $\|c\|_0 < f(x^*)$ (with such c and f fixed) and a is sufficiently large so that $b = ac$ is sufficiently large. Then $p \approx \sigma = e^{-\bar{b}} \approx 0$ and $f(\sigma x^*) \approx f'(0)\sigma x^*$, and so (note that $\bar{a} \le \bar{b}/c_1$)

$$\sigma p f(\sigma x^*)\bar{a} \le \frac{\sigma p f(\sigma x^*)\bar{b}}{c_1} \approx \frac{\bar{b}}{e^{3\bar{b}}} \cdot \frac{f'(0)x^*}{c_1} \approx 0.$$

Hence, if b is sufficiently large, then the assumption in Theorem 3.1 cannot be satisfied and no existence result can be obtained for (1.2) from Theorem 3.1 . However, since $\|c\|_0 < f(x^*)$, Theorem 2.1 yields that (1.2) has at least two positive ω-periodic solutions for sufficiently small r.

Theorem 3.1 will be derived from Theorem 3.3, whose proof is based on Theorem 3.4 ([7]).

Theorem 3.3. *Let g, r_0, b, $\bar{a}$, $\bar{b}$, σ and p be the same as in Theorem 3.1, and $q = p/\sigma$. Let $0 < R_1 < R_2 < x^* < R_3 < R_4 < \gamma$ be real numbers.*

(i) If $qf(R_1)\bar{a} \le 1$ and $\sigma p f(\sigma R_2)\bar{a} \ge 1$, then the equation (1.2) has a solution $x_1 \in C_\omega$ with $\sigma R_1 \le x_1 \le R_2$.

(ii) Let $\tilde{R}_3 \in (0, x^*)$ such that $f(\tilde{R}_3) = f(R_3)$. Let

$$L_1 := \begin{cases} f(R_3), & \text{if } \sigma R_3 \ge \tilde{R}_3, \\ f(\sigma R_3), & \text{if } \sigma R_3 < \tilde{R}_3, \end{cases} \qquad L_2 = \begin{cases} f(\sigma R_4), & \text{if } \sigma R_4 \ge x^*, \\ f(x^*), & \text{if } \sigma R_4 < x^*. \end{cases}$$

If $\sigma p L_1 \bar{a} \ge 1$ and $qL_2\bar{a} \le 1$, then (1.2) has a solution $x_2 \in C_\omega$ with $\sigma R_3 \le x_2 \le R_4$.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space, $\mathcal{K}$ be a cone in X, and Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. Let $\Phi : \mathcal{K} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ be a completely continuous operator such that either*

(i) $\|\Phi x\| \le \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_1$ and $\|\Phi x\| \ge \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_2$; or

(ii) $\|\Phi x\| \ge \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_1$ and $\|\Phi x\| \le \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_2$.

Then Φ has a fixed point in $\mathcal{K} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Recall that K is a cone in X if $\mathcal{K}$ is a convex and closed nonempty subset of X satisfying: (i) $x = 0$ if x and $-x$ are both in $\mathcal{K}$, and (ii) $\beta_1 x_1 + \beta_2 x_2 \in K$ if x_1 and x_2 are in $\mathcal{K}$ and $\beta_1 \ge 0$ and $\beta_2 \ge 0$.

Proof of Theorem 3.3. We first note that $x \in C_\omega$ is a solutions of (1.2) if and only if it is a solution in C_ω of the integral equation

$$x(t) = \int_t^{t+\omega} G(t,s)a(s)x(s)f(x(s - r(s)))\, ds,$$

where G is the Green function given by

$$G(t,s) = \frac{e^{\int_t^s b(\tau)\, d\tau}}{e^{\int_0^\omega b(\tau)\, d\tau} - 1}.$$

Since $\int_t^{t+\omega} b(s)\, ds = \int_0^\omega b(s)\, ds$, it follows that, for $0 \le t \le \omega$ and $t \le s \le t + \omega$,

$$p = \frac{1}{e^{\int_0^\omega b(\tau)\, d\tau} - 1} \le G(t,s) \le \frac{e^{\int_0^\omega b(\tau)\, d\tau}}{e^{\int_0^\omega b(\tau)\, d\tau} - 1} = q.$$

We define a cone $\mathcal{K} = \{x \in C_\omega : \min_{t \in [0,\omega]} x(t) \ge \sigma\|x\|_0\}$, and an operator $\Phi : \mathcal{K} \to C_\omega$ by

$$(\Phi x)(t) = \int_t^{t+\omega} G(t,s)a(s)x(s)f(x(s - r(s)))\, ds.$$

Then, we have, for $x \in \mathcal{K}$,

$$|(\Phi x)(t)| \leq q \int_t^{t+\omega} a(s)x(s)f(x(s-r(s)))\,ds = q \int_0^{\omega} a(s)x(s)f(x(s-r(s)))\,ds,$$

and so,

$$\begin{aligned}(\Phi x)(t) &\geq p \int_t^{t+\omega} a(s)x(s)f(x(s-r(s)))\,ds = \sigma q \int_0^{\omega} a(s)x(s)f(x(s-r(s)))\,ds \\ &\geq \sigma \|\Phi x\|_0.\end{aligned}$$

This shows that Φ maps $\mathcal{K}$ into $\mathcal{K}$. It is easily shown that Φ is completely continuous on $\mathcal{K}$ (see Appendix).

By the property of f and the assumption in (i) of Theorem 3.3, we have that if x is in $\mathcal{K}$ and $\|x\|_0 = R_1$ then $|(\Phi x)(t)| \leq qR_1 f(R_1)\bar{a} \leq R_1$, and so $\|\Phi x\|_0 \leq R_1$; if x is in $\mathcal{K}$ and $\|x\|_0 = R_2$ then $|(\Phi x)(t)| \geq \sigma p R_2 f(\sigma R_2)\bar{a} > R_2$ and so $\|\Phi x\|_0 \geq R_2$. Similarly, we have $\|\Phi x\|_0 \geq \sigma p R_3 L_1 \bar{a} \geq R_3$ for $x \in \mathcal{K}$ and $\|x\|_0 = R_3$, and $\|\Phi x\|_0 \leq qR_4 L_2 \bar{a} \leq R_4$ for $x \in \mathcal{K}$ and $\|x\|_0 = R_4$. Therefore, the conclusions for x_1 and x_2 in Theorem 3.3 follow from Theorem 3.4 directly with $\Omega_1 = B(R_1)$ and $\Omega_2 = B(R_2)$, and $\Omega_1 = B(R_3)$ and $\Omega_2 = B(R_4)$ respectively, where $B(R_i)$ $(i = 1, 2, 3, 4)$ are the open balls of the center 0 and the radius R_i in C_ω.

Proof of Theorem 3.1. In Theorem 3.3, if we take R_2 and R_3 are sufficiently close to x^* and R_1 and R_4 sufficiently close to 0 and $\gamma = \infty$ respectively, then we have $\tilde{R}_3 \sim R_3$ and $\sigma R_4 \sim \infty$ and so $\sigma R_3 < \tilde{R}_3$ and $\sigma R_4 \geq x^*$ and so $L_1 = f(\sigma R_3) \sim f(\sigma x^*)$ and $L_2 = f(\sigma R_4) \sim 0$. Thus, if $\sigma p f(\sigma x^*)\bar{a} \geq 1$, then the conditions in Theorem 3.3 are satisfied with the above choice of R_i $(i = 1, 2, 3, 4)$, yielding the existence of x_1 and x_2 with $0 < \sigma R_1 \leq x_1 \leq R_2 < x^*$, $\sigma x^* < \sigma R_3 \leq x_2 \leq R_4 < \infty$, and $\|x\|_0 \geq R_3 > x^*$. It follows from the remark at the end of Section 1 that $\|x_1\|_0 \geq d_1$. This shows Theorem 3.1.

REMARK 3.5. *(i) Theorem 3.1 and Theorem 3.3 hold (without any changes) for the equation*

$$x' = a(t)x\Big[f\Big(x\Big(t - r(t, x(t))\Big)\Big) - c(t)\Big], \tag{3.1}$$

where r is continuous and ω-periodic about its first argument t.

(ii) For the general situation where $g(t,x) \neq x$ and $r_0 \neq 0$ in (1.2), we can write (1.2) as

$$x' = -\Big[a(t)c(t)\frac{g(t, x(t - r_0(t)))}{x}\Big]x + \Big[a(t)f(x(t - r(t)))\frac{g(t, x(t - r_0(t)))}{x}\Big]x.$$

A modification of the proof of Theorem 3.3 will yield a sufficient condition for (1.2) to have at least two positive ω-periodic solutions. However, the definitions for σ, p and q are more involved and thus the statement of the result is omitted.

4. **Summary.**

We have obtained two sets of sufficient conditions for (1.2) to have at least two periodic solutions x_1 and x_2. Those conditions are easily verified. We also give the estimates for the locations of x_1 and x_2 in our theorems. Generally speaking, it is more difficult to discuss the stability and local uniqueness of these solutions due to the appearance of delays in (1.2) (see [2, 3, 8, 11, 12]). If $r_0 \equiv 0$ and $r \equiv 0$, then the directional field analysis for (1.2) shows that x_1 is unstable and x_2 is stable. We expect that this conclusion still holds for sufficiently small r_0 and

r. Corollary 2.4 implies that x_1 and x_2 are locally unique in the case where c is a constant function. We shall explore those topics further in future work.

When applying our existence results to the model equation (1.1), it yields that, under the certain parameter regions of the model, the number $x(t)$ of sexually mature worms may change periodically. Such periodic behavior is certainly expected biologically due to the periodicity of the environment of the model (e. g., seasonal effects of weather, food supplies, mating habits, etc.). The stable periodic solution may provide a global attractor for (1.1) which is important for the study of the dynamics of the model problem. We hope that the parameter regions obtained from our results are biologically realistic.

Appendix. In this appendix, we shall show that the mapping T defined in the proof of Theorem 2.1 is continuous and compact on $\mathcal{S}$, so is Φ defined in the proof of Theorem 3.3.

We first show that $T : \mathcal{S} \to \mathcal{S}$ is continuous. Given $x_0 \in \mathcal{S}$, let $\{x_n\}_{n=1}^{\infty} \subset \mathcal{S}$ be a sequence such that $\|x_n - x_0\|_0 \to 0$ as $n \to \infty$. Let $y_0 = Tx_0$ and $y_n = Tx_n$. We have $y_0, y_n \in \mathcal{S}$. By the right-hand side of the equation (2.2), we see that $(y_n)'$ $(n = 1, 2, \cdots)$ are uniformly bounded and so y_n are equicontinuous. Note that $T(\mathcal{S}) \subset \mathcal{S}$ is bounded in C_ω. Hence, by Arzela-Ascoli's theorem there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ such that y_{n_k} is uniformly convergent to $y^* \in \mathcal{S}$ on $[0, \omega]$ as $k \to \infty$ (i.e., $\|y_{n_k} - y^*\|_0 \to 0$ as $k \to \infty$). Since $\|x_n - x_0\|_0 \to 0$ as $n \to \infty$, it follows that y^* satisfies:

$$y'(t) = a(t)g(t, x_0(t - r_0(t)))\Big[f(y(t)) - \Big(c(t) + f(x_0(t)) - f(x_0(t - r(t)))\Big)\Big]. \quad (4.1)$$

Note that y_0 is also a solution of (4.1) in $\mathcal{S}$. Thus, the uniqueness of the solution of (4.1) in $\mathcal{S}$ (from Lemma 2.3) implies that $y^* = y_0$. This uniqueness also implies that the whole sequence y_n goes to y_0 uniformly. This shows that T is continuous at x_0, thereby on $\mathcal{S}$.

Since $\mathcal{S}$ is bounded, $(T(\mathcal{S}))'$ is bounded in C_ω from the right-hand side of (2.2) and so $T(\mathcal{S})$ is equicontinuous. Therefore, by Arzela-Ascoli's lemma $T(\mathcal{S})$ is compact, and so T is compact. This shows that T is continuous and compact on $\mathcal{S}$.

We next show that Φ is continuous and compact on $\Omega := \mathcal{K} \cap (B(R_2) \setminus B(R_1))$. Given $x_0 \in \Omega$, let $\{x_n\}_{n=1}^{\infty} \subset \Omega$ such that $\|x_n - x_0\|_0 \to 0$ as $n \to \infty$. Since $xf(x)$ is uniformly continuous on $[-R_2, R_2]$ and $G(t, s)a(s)$ is bounded for $t, s \in [0, \omega]$, it follows from the definition of Φ that $\|\Phi x_n - \Phi x_0\|_0 \to 0$ as $n \to \infty$. This shows that Φ is continuous at x_0, thereby continuous on Ω. Note that, for $x \in \Omega$,

$$(\Phi x)' = a(t)x(t)[f(x(t - r(t)) - c(t)].$$

It follows that $\|(\Phi x)'\|_0$ is uniformly bounded on Ω . Note that $\Phi(\Omega) \subset \Omega$. It follows from Arzela-Ascoli's lemma that $\Phi(\Omega)$ is a compact set and so Φ is a compact operator on Ω. The similar conclusion holds for Φ on $\mathcal{K} \cap (B(R_4) \setminus B(R_3))$.

Acknowledgments. The author thanks Professor Wenzhang Huang and a referee for their valuable suggestions.

REFERENCES

[1] H.I. Freedman and J. Wu, *Periodic solutions of single-species models with periodic delay,* SIAM J. Math. Anal., 23 (1992), 689–701.

[2] I. Gyori and G. Ladas, "Oscillation Theory of Delay Differential Equations", Clarendon Press, Oxford, 1991.

[3] J.K. Hale, "Theory of Functional Differential Equations", Springer-Verlag, Berlin, 1977.
[4] J.K. Hale, L.T. Magalhaes and M.W. Oliva, "An introduction to infinite dimensional systems–geometric theory," Springer-Verlag, New York, 1984.
[5] J. Kurzweil, *Small delays don't matter,* in "Lecture Notes in Math.", Vol. 206, 47–49, Springer-Verlag, New York, 1971.
[6] D. Jiang, J. Wei, and B. Zhang, *Positive periodic solutions of functional differential equations and population models,* Elec. J. Diff. Eqns., 71 (2002), 1–13.
[7] M.A. Krasnosel'skii, "Positive Solutions of Operator Equations", Noordhoff, 1964.
[8] Y. Kuang, "Delay Differential Equations with Applications in Population Dynamics", Academic Press, New York, 1993.
[9] Y. Li and Y. Kuang, *Periodic solutions in periodic state-dependent delay equations and populations models,* Proc. Amer. Math. Soc., 130 (2001), 1345–1353.
[10] R. May and R. M. Anderson, "Infectious Diseases of Humans: Dynamics and Control", Oxford Science Publications, 1995.
[11] R. Ouifki, M.L. Hbid and O. Arino, *Attractiveness and Hopf bifurcation for retarded differential equations,* Commun. Pure Appl. Anal., 2 (2003), 147–158.
[12] S.H. Saker, *On a delay host macroparasite model with periodic coefficients,* preprint.
[13] G.F. Webb, *The prime number periodical cicada problem,* Discrete Contin. Dyn. Syst. Ser. B, 1 (2001), 387–399.

Received May 2003; revised August 2003; final version January 2004.

E-mail address: ais@email.uah.edu