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GLOBAL STABILITY OF EQUILIBRIA IN A TICK-BORNE DISEASE MODEL

Shangbing Ai
Department of Mathematical Sciences
University of Alabama in Huntsville, Huntsville, AL 35899, USA

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Abstract. In this short note we establish global stability results for a four-dimensional nonlinear system that was developed in modeling a tick-borne disease by H.D. Gaff and L.J. Gross (Bull. Math. Biol., **69** (2007), 265–288) where local stability results were obtained. These results provide the parameter ranges for controlling long-term population and disease dynamics.

1. **Introduction.**

In the United States, ticks are the most common vectors of vector-borne diseases. Ticks can carry and transmit a remarkable array of pathogens, such as bacteria, spirochetes, rickettsiae, protozoa, viruses, nematodes, and toxins, and cause several human diseases including Lyme disease, Rocky Mountain spotted fever, human babesiosis, ehrlichiosis, tick-borne relapsing fever, Colorado tick fever and tick paralysis. The spatial and temporal patterns of outbreaks of these diseases tend to be erratic. Recent increases in reported outbreaks of these diseases have led to increased interest in understanding and controlling epidemics involving these transmission vectors. Various mathematical models have been developed to achieve such goals. See references [1], [3], [5], [6], [7] and [8] for some of these models.

In this paper we study a model that was proposed in [4] for the dynamics of tick-borne infection in the case of a single host, a single pathogen, and a single life stage. The model is used to describe the interaction of lone-star ticks and their hosts, the white-tailed deer, as their populations become infected with the E. chaffeensis rickettsia. It is assumed in the model that the population sizes for both hosts and ticks are nonconstant. The governing equations for the model are

$$\begin{aligned}
\frac{dN}{dt} &= \beta(\frac{K-N}{K})N - bN,\\
\frac{dV}{dt} &= \hat{\beta}(\frac{MN-V}{MN})V - \hat{b}V,\\
\frac{dX}{dt} &= \hat{A}(\frac{Y}{N})(V-X) - \hat{\beta}\frac{VX}{MN} - \hat{b}X,\\
\frac{dY}{dt} &= A(\frac{N-Y}{N})X - \beta\frac{NY}{K} - (b+\nu)Y,
\end{aligned} \tag{1}$$

where N and V are the host and tick population densities, Y and X are the densities of individuals in host and tick populations that are infected with the disease respectively, and β, $\hat{\beta}$, b, $\hat{b}$, K, M, A, $\hat{A}$ and ν are nonnegative parameters and their

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physical meanings are as follows: β and $\hat{\beta}$ are the growth rates of hosts and ticks respectively, b and $\hat{b}$ are the external death rates of hosts and ticks respectively, K is the carrying capacity for hosts per m^2, M is the maximum number of ticks per host, A is the transmission date from hosts to ticks, $\hat{A}$ is the transmission rate from ticks to hosts, ν is the recovery rate of hosts. Two sets of numerical estimates for these parameters are given in [4]. For a further description of the model we refer the interested reader to [4] and the references therein.

In order to understand the long-term population and disease dynamics and help determine the expected results of different control programs, Gaff and Gross [4] performed equilibrium analysis for the system (1). Using the scalings

$$\begin{aligned}
&n = \frac{K}{N}, \quad v = \frac{V}{MK}, \quad y = \frac{Y}{K}, \quad x = \frac{X}{KM}, \quad \tilde{t} = \beta t, \\
&s_0 = 1 - \frac{b}{\beta}, \quad s_1 = \frac{\hat{\beta}}{\beta}, \quad s_2 = \frac{\hat{b}}{\hat{\beta}}, \quad s_3 = \frac{AM}{\beta}, \quad s_4 = \frac{b+\nu}{\beta}, \quad s_5 = \frac{\hat{A}}{\beta},
\end{aligned}$$

they first transformed (1) into the following nondimensional system with six parameters rather than the original nine (after dropping the tilde from $\tilde{t}$):

$$\begin{aligned}
\frac{dn}{dt} &= (s_0 - n)n, \\
\frac{dv}{dt} &= s_1 v\Big(1 - s_2 - \frac{v}{n}\Big), \\
\frac{dx}{dt} &= s_5(v - x)\frac{y}{n} - s_1 s_2 x - s_1 \frac{vx}{n}, \\
\frac{dy}{dt} &= s_3\Big(1 - \frac{y}{n}\Big)x - s_4 y - yn,
\end{aligned} \tag{2}$$

where the parameters s_0, s_1, s_2, s_3, s_4, and s_5 are non-negative. Since only the region $\mathcal{P}$ given by

$$\mathcal{P} := \{(n, v, x, y) : n > 0, v > 0, x \geq 0, y \geq 0\}$$

is biologically relevant to the system (2), they then showed that if s_0, s_1, s_3, and s_5 are positive, then $\mathcal{P}$ is positively invariant for the flows of (2). Here is an alternative way to show this property. Let (n_0, v_0, x_0, y_0) be an arbitrary point in $\mathcal{P}$ and $(n(t), v(t), x(t), y(t))$ be the solution of (2) through this point at $t = 0$, whose right maximal interval of existence is $[0, \beta)$ where $0 < \beta \leq \infty$. Then, we have following expressions for $t \in [0, \beta)$:

$$\begin{aligned}
n(t) &= n_0 e^{\int_0^t [s_0 - n(\tau)]\, d\tau}, \\
v(t) &= v_0 e^{\int_0^t s_1[1 - s_2 - v(\tau)/n(\tau)]\, d\tau}, \\
x(t) &= x_0 e^{\int_0^t a(\tau)\, d\tau} + s_5 \int_0^t e^{\int_\tau^t a(s)\, ds}[v(\tau)y(\tau)/n(\tau)]\, d\tau, \\
y(t) &= y_0 e^{\int_0^t b(\tau)\, d\tau} + s_3 \int_0^t e^{\int_\tau^t b(s)\, ds} x(\tau)\, d\tau,
\end{aligned} \tag{3}$$

where

$$a(t) := -s_5 \frac{y(t)}{n(t)} - s_1 s_2 - s_1 \frac{v(t)}{n(t)}, \qquad b(t) := -s_3 \frac{x(t)}{n(t)} - s_4 - n(t).$$

It follows from (3) that $n(t) > 0$, $v(t) > 0$, $x(t) > 0$ and $y(t) > 0$ for all $t \in (0, \beta)$, i.e., $(n(t), v(t), x(t), y(t))$ is in the interior of $\mathcal{P}$ for all $t \in (0, \beta)$.

From [4], the system (2) has three equilibria:

$$E_1 = (s_0, 0, 0, 0), \quad E_2 = (s_0, s_0(1-s_2), 0, 0), \quad E_3 = (s_0, s_0(1-s_2), x^*, y^*),$$

where

$$x^* = \frac{s_0 R_0}{s_3(s_1+s_5)}, \quad y^* = \frac{s_0 R_0}{s_5[s_3(1-s_2)+s_4+s_0]}, \quad R_0 = s_3 s_5(1-s_2) - s_1(s_0+s_4).$$

By analyzing the eigenvalues of the variational systems of (2) at E_1, E_2 and E_3, Gaff and Gross proved the following result on the local stability of these equilibria, in the Appendix of [4]:

Theorem 1.1. *Assume that $s_i > 0$ for $i = 0, \cdots, 5$.*

(i) If $s_2 > 1$, then E_1 is a locally asymptotically stable solution of (2).

(ii) If $s_2 < 1$ and $R_0 \le 0$, then E_2 is a locally asymptotically stable solution of (2).

(iii) If $s_2 < 1$ and $R_0 > 0$, then E_3 is a locally asymptotically stable solution of (2).

At the end of the reference [4], based on phase plane diagrams in the xy-plane and numerical results Gaff and Gross conjectured that E_3 is globally stable under the hypotheses of Theorem 1.1 (iii). The purpose of this note is to confirm this conjecture. Indeed, we prove the following:

Theorem 1.2. *Assume that $s_i > 0$ for $i = 0, \cdots, 5$. Then under the assumptions in Theorem 1.1 (i), (ii), (iii), E_1, E_2, E_3 are globally asymptotically stable in $\mathcal{P}$ for the system (2), respectively.*

Remark 1. (a) Our proof implies that (i) holds also for $s_2 = 1$.

(b) In case (i), E_1 is the only equilibrium point of (2) lying in $\bar{\mathcal{P}}$ (the closure of $\mathcal{P}$). In case (ii), E_1 and E_2 are the equilibria in $\bar{\mathcal{P}}$. In case (iii), E_1 and E_3 are the equilibria in $\bar{\mathcal{P}}$.

The proof of Theorem 1.2 is given in the next section.

2. **Proof of Theorem 1.2.** In order to prove Theorem 1.2 we need the following lemma.

Lemma 2.1. *Assume that $s_i > 0$ for $i = 0, \cdots, 5$. Then*

(i) The set

$$\mathcal{Q} := \mathcal{P} \cap \{(n, v, x, y) : n \le 2s_0,\ v \le 3s_0|1-s_2|,\ x \le 3s_0|1-s_2|,\ y \le 2s_0\}$$

is positively invariant for (2).

(ii) Every solution of (2) starting in $\mathcal{P}$ enters $\mathcal{Q}$ eventually.

Proof. Since $\mathcal{P}$ is positively invariant, it suffices to show in (i) that the vector fields of (2) point inside of $\mathcal{Q}$ on its boundaries: $n = 2s_0$, or $v = 3s_0|1-s_2|$, or $x = 3s_0|1-s_2|$, or $y = 2s_0$. This assertion follows from easy and direct verifications.

It remains to show (ii). We just prove the case that $s_2 < 1$ since the case $s_2 > 1$ can be prove similarly. To this end, let (n_0, v_0, x_0, y_0) be an arbitrary point in $(\mathcal{P} \backslash \mathcal{Q})$, and $\phi(t) = (n(t), v(t), x(t), y(t))$ be the solution of (2) through this point at $t = 0$ with $t \in [0, \beta)$ where $[0, \beta)$ is the right maximal interval of existence of $\phi(t)$. We first show that $\beta = \infty$. The first equation of (2) for n is a logistic equation and can be solved explicitly, in particular n is increasing if $n_0 \in (0, s_0)$, $n \equiv s_0$ if $n_0 = s_0$ and decreasing if $n > s_0$. We thus have $n(t) \le$

$N_0 := \max\{s_0, n_0\}$ for $t \in [0, \beta)$. Inserting this estimate into the second equation of (2) yields $v'(t) < s_1 v(t)(1 - s_2 - v(t)/N_0)$ for $t \in [0, \beta)$, which implies that $v(t) \leq V_0 := \max\{v_0, N_0(1 - s_2)\}$ for $t \in [0, \beta)$. Then, from the third and fourth equations of (2) we get $x'(t) < s_5(V_0 - x(t))y(t)/n(t)$ and $y'(t) < s_3(1 - y(t)/N_0)x(t)$ for $t \in [0, \beta)$, which together with $x(t) > 0$ and $y(t) > 0$ for $t \in (0, \beta)$ yield $x(t) < X_0 := \max\{x_0, V_0\}$ and $y(t) < Y_0 := \max\{y_0, N_0\}$ for $t \in [0, \beta)$, respectively. From these estimates we conclude that $\beta = \infty$.

In the rest of the proof we show that ϕ enters $\mathcal{Q}$ after a certain time. It follows from the first equation that $\lim_{t\to\infty} n(t) = s_0$, and hence there is a $t_0 > 0$ such that $n(t) < 2s_0$ for $t \geq t_0$. Then from the second equation of (2) we get $v'(t) < s_1 v(t)[1 - s_2 - v(t)/(2s_0)]$ for $t \in [t_0, \infty)$ which yield that there is a $t_1 \geq t_0$ such that $v(t_1) < 3s_0(1 - s_2)$ for if such a t_1 did not exist we would have $v'(t) < s_1[1 - s_2 - (3/2)(1 - s_2)]v(t) = (3/2)s_1(1 - s_2)^2 v(t)$ for $t \geq t_0$ so that $v(t) \to -\infty$ as $t \to \infty$. Since $s_1 v[1 - s_2 - v/(2s_0)] < 0$ whenever $v = 3s_0(1 - s_2)$, it follows that $v(t) < 3s_0(1 - s_2)$ for $t \geq t_1$. Now, for $t \geq t_1$, we have $x'(t) < s_5[3s_0(1 - s_2) - x(t))]y(t)/n(t) - s_1 s_2 x(t)$. This implies that there is a $t_2 \geq t_1$ such that $x(t_2) \leq 3s_0(1 - s_2)$ for if such a t_2 did not exist we would have $x(t) > 3s_0(1 - s_2)$ for all $t \geq t_1$ so that $x'(t) < -s_1 s_2 x(t)$ and so $x(t) \to 0$ as $t \to \infty$, a contradiction. Since $s_5[3s_0(1 - s_2) - x)](y/n) - s_1 s_2 x < 0$ whenever $x = 3s_0(1 - s_2)$, it follows that $x(t) < 3s_0(1 - s_2)$ for $t \geq t_2$. Finally, noting that $y'(t) < [1 - y(t)/(2s_0)]x(t) - s_4 y(t)$ for $t \geq t_2$, in a similar manner to that for x we have that there is a $t_3 > t_2$ such that $y(t) \leq 2s_0$ for $t \geq t_3$. Thus, we conclude that $\phi(t) \in \mathcal{Q}$ for $t \geq t_3$. This completes the proof of Lemma 2.1. □

Proof of Theorem 1.2. Let $P_0 = (n_0, v_0, x_0, y_0)$ be an arbitrary point in $\mathcal{P}$ and $\phi(t) = (n(t), v(t), x(t), y(t))$ be the solution of (2) through this point at $t = 0$. We have from Lemma 2.1 and the general structure of an ω-limit set (cf. [2]) that $\phi(t)$ is defined for $t \in [0, \infty)$ and its ω-limit set, denoted by $\omega(P_0)$, is nonempty, compact, connected, composed of entire orbits of (2), and $\omega(P_0) \subseteq \mathcal{Q}$. We need to show that $\omega(P_0) = \{E_i\}$ $(i = 1, 2, 3)$ in cases (i), (ii), (iii), respectively. The first two equations of (2) are both Bernouli's equations so that we have, for $t > 0$,

$$\begin{aligned}
\frac{1}{n(t)} &= \frac{1}{n_0}e^{-s_0 t} + \frac{1}{s_0}(1 - e^{-s_0 t}),\\
\frac{1}{v(t)} &= \frac{1}{v_0}e^{-s_1(1-s_2)t} + s_1 \int_0^t e^{-s_1(1-s_2)(t-\tau)} \frac{1}{n(\tau)}\, d\tau,
\end{aligned}$$

from which we get directly $\lim_{t\to\infty}(n(t), v(t)) = (s_0, v_\infty)$ where $v_\infty = \max\{0, s_0(1 - s_2)\}$. This implies that $\omega(P_0)$ lies on the invariant plane $n = s_0$ and $v = v_\infty$ in the four-dimensional phase space of (2). Let $(s_0, v_\infty, X_0(t), Y_0(t))$ be an arbitrary solution of (2) in $\omega(P_0)$ defined on $(-\infty, \infty)$. Then, $(X_0(t), Y_0(t))$ is a solution of the planar system

$$\begin{aligned}
\frac{dX}{dt} &= f_1(X, Y) := \frac{s_5}{s_0}Y(v_\infty - X) - s_1 s_2 X - \frac{s_1 v_\infty}{s_0}X,\\
\frac{dY}{dt} &= f_2(X, Y) := s_3(1 - \frac{Y}{s_0})X - (s_4 + s_0)Y.
\end{aligned} \tag{4}$$

Note that

$$\frac{\partial f_1}{\partial X} + \frac{\partial f_2}{\partial Y} = [-s_1 s_2 - \frac{s_1 v_\infty}{s_0} - \frac{s_5}{s_0}Y] + [-(s_4 + s_0) - \frac{s_3}{s_0}X] < 0.$$

It follows from Bendixson's Criterion that (4) does not have any periodic solutions and homoclinic loops in the first quadrant. This rules out $(X_0(t), Y_0(t))$ as a periodic or homoclinic solution. Therefore, $(X_0(t), Y_0(t))$ is either an equilibrium solution or a heteroclinic solution.

Under the assumptions in cases (i) and (ii), we have that $(0, 0)$ is the only equilibrium point of (4) in the first quadrant in the xy-plane. This implies that (4) does not have any heterolinic solutions. We thus have $(X_0(t), Y_0(t)) = (0, 0)$ for all $t \in \mathbb{R}$. Therefore, we conclude $\omega(P_0) = \{E_1\}$ and $\omega(P_0) = \{E_2\}$ in (i) and (ii), respectively.

Under the assumptions in (iii), we have that (4) has only two equilibria $(0, 0)$, (x^*, y^*) in the first quadrant of the XY-plane. We claim that (4) has a unique heteroclinic orbit from $(0, 0)$ to (x^*, y^*) lying in the first quadrant. To see this, we need to study the local property of (4) near $(0, 0)$. It is easy to find that the linearized system of (4) at $(0, 0)$ is:

$$\begin{aligned} \frac{dX}{dt} &= -s_1 X + s_5(1 - s_2)Y, \\ \frac{dY}{dt} &= s_3 X - (s_0 + s_4)Y, \end{aligned} \tag{5}$$

with two eigenvalues given by

$$\lambda_{\pm} = \frac{1}{2}\Big[-(s_0 + s_1 + s_4) \pm \sqrt{(s_0 + s_1 + s_4)^2 + 4R_0}\Big].$$

Since $R_0 > 1$ by assumption, we see that $(0, 0)$ is a saddle equilibrium point of (5). A simple computation shows that the eigenvectors of (5) associated to $\lambda_{\pm}$ are $(1, b_{\pm})^{\top}$ where

$$b_{\pm} = \frac{s_1 + \lambda_{\pm}}{s_5(1 - s_2)}.$$

It is clear that $b_+ > 0$. Since

$$s_1 + \lambda_- = \frac{1}{2}\Big[(s_1 - s_0 - s_4) - \sqrt{(s_1 - s_0 - s_4)^2 + 4s_3 s_5(1 - s_2)}\Big],$$

it follows that $b_- < 0$. By the stable manifold theorem, the system (4) has a one-dimensional stable manifold at $(0, 0)$ that lies in the second and fourth quadrants and one-dimensional unstable manifold that lies in first and third quadrants. Using the Poincare-Bendixson theorem and the fact established above (i.e., (4) has neither periodic orbit nor homoclinic loops in the first quadrant) we conclude that the branch of this unstable manifold in the first quadrant approaches (x^*, y^*) as $t \to \infty$, which gives a heteroclinic orbit of (4). The dimension of the unstable manifold of (4) at $(0, 0)$ implies that this is the only heteroclinic orbit of (4).

Now we claim that $(X_0(t), Y_0(t))$ cannot be this heteroclinic orbit. Suppose that this is false. Since E_3 is a locally asymptotically stable solution of (2), for a given sufficiently small ε with $0 < \varepsilon < (1/2)\sqrt{(x^*)^2 + (y^*)^2}$ there is a $\delta > 0$ such that any solution of (2) starting in $B(E_3, \delta)$ stays in $B(E_3, \varepsilon)$ for all $t \geq 0$. Since $(X_0(t), Y_0(t)) \to (x^*, y^*)$ as $t \to \infty$, it follows that there is a $t_0 > 0$ sufficiently large such that $(s_0, v_\infty, X_0(t_0), Y_0(t_0)) \in B(E_3, \delta)$. Here $B(E_3, \delta)$ and $B(E_3, \varepsilon)$ denote the open balls in the phase space of (2) centered at E_3 with radii of δ and ε, respectively. Then, by virtue of $(s_0, v_\infty, X_0(t_0), Y_0(t_0)) \in \omega(P_0)$, there is a $t_1 > t_0$ sufficiently large such that $\phi(t_1) \in B(E_3, \delta)$, and thus $\phi(t) \in B(E_3, \varepsilon)$ for any $t \geq t_1$. This together with $(X_0(-\infty), Y_0(-\infty)) = (0, 0)$ implies that the points

$(s_0, v_\infty, X_0(t), Y_0(t))$ cannot be in $\omega(P_0)$ for sufficiently negative t, a contradiction. This shows the above claim.

Therefore, we conclude that either $(X_0(t), Y_0(t)) \equiv (0, 0)$ or $(X_0(t), Y_0(t)) \equiv (x^*, y^*)$. This yields that $\omega(P_0) \subseteq \{E_2, E_3\}$. Then using the fact that $\omega(P_0)$ is connected, we obtain that either $\omega(P_0) = \{E_2\}$ or $\omega(P_0) = \{E_3\}$. The first alternative is impossible since for otherwise we would have from the stable manifold theorem

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c \left[\begin{pmatrix} 1 \\ b_- \end{pmatrix} + o(1) \right] e^{\lambda_- t} \qquad \text{as} \quad t \to \infty$$

for some constant $c \neq 0$, which yields $x(t)y(t) < 0$ for sufficiently large t, contradicting that $x(t) > 0$ and $y(t) > 0$ for all $t > 0$. Hence, it must be the case that $\omega(P_0) = \{E_3\}$, thereby completing the proof of Theorem 1.2. $\square$

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REFERENCES

[1] T. E. Awerbuch and S. Sandberg, *Trends and oscillations in tick population dynamics*, J. Theor. Biol., **175** (1995), 511–516.
[2] E. Coddington and N. Levinson, “ *Ordinary Differential Equations*,” New York: McGraw-Hill, 1955.
[3] W. E. Fitzgibbon, M. E. Parrott, and G. F. Webb, *A diffusive epidemic model for a host-vector system*, in “Differential Equations and Applications to Biology and Industry” (eds. M. Martelli, K. Cooke, E. Cumberbatch, B. Tang, H. Thieme), Singapore: World Scientific Press, (1996), 401–408.
[4] H. D. Gaff and L. J. Gross, *Modelling tick-borne disease: A metapopulation model*, Bull. Math. Biol., **69** (2007), 265–288.
[5] M. Ghosh and A. Pugliese, *Seasonal population dynamics of ticks, and its influence on infection transmission: A semi-discrete approach*, Bull. Math. Biol., **66** (2004), 1659–1684.
[6] G. A. Mount, D. G. Haile, and E. Daniels, *Simulation of blacklegged tick (Acari: Ixodidea) population dynamics and transmission of borrelia burgdorferi*, J. Med. Entomol., **34** (1997), 461–484.
[7] G. A. Mount, D. G. Haile, and E. Daniels, *Simulation of management strategies for the blacklegged tick (Acari: Ixodidea) and the Lyme disease spirochete, borrelia burgdorferi*, J. Med. Entomol., **90** (1997), 672–683.
[8] S. Randolph, *Epidemiological uses of a population model for the tick Rhipicephalus appendiculatus*, Trop. Med. Int. Health, **4** (1999), A34–A42.

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E-mail address: ais@email.uah.edu