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Dynamics of a Canonical Electrostatic MEMS/NEMS System

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The mass-spring model of electrostatically actuated microelectromechanical systems (MEMS) or nanoelectromechanical systems (NEMS) is pervasive in the MEMS and NEMS literature. Nonetheless a rigorous analysis of this model does not exist. Here periodic solutions of the canonical mass-spring model in the viscosity dominated time harmonic regime are studied. Ranges of the dimensionless average applied voltage and dimensionless frequency of voltage variation are delineated such that periodic solutions exist. Parameter ranges where such solutions fail to exist are identified; this provides a dynamic analog to the static "pull-in" instability well known to MEMS/NEMS researchers.

KEY WORDS: MEMS; nanotechnology; electrostatics; periodic solutions; saddle-node bifurcation; shooting method.

MATHEMATICS SUBJECT CLASSIFICATION (1991): 34C15; 34C60; 70K40.

1. INTRODUCTION

As the characteristic length of engineering systems approaches the micro or nanometer scale the role of electrostatics grows correspondingly. Often perceived as a nuisance in the macro-world, such as in the case of the destruction of sensitive electronic circuits due to electrostatic discharge (ESD),[4] electrostatic forces are increasingly being used to provide accurate,

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[4]ESD is no laughing matter. Before the advent of nonflammable anesthetics an errant spark from a doctor's scalpel would sometimes ignite the ether in a patient's lungs, rendering the operation a failure.

controlled, stable locomotion for micro and nanoelectromechancical systems (MEMS or NEMS). In this approach, voltage differences are applied between mechanical components of the system. This induces a Coulomb force between components which is varied in strength by varying the applied voltage. This technique is already employed in devices such as accelerometers [4], optical switches [5], microgrippers [7], micro force gauges [19], transducers [3], micro pumps [16] and nanotweezers [12].

In order to understand the operation of such devices researchers in the MEMS and NEMS communities have relied upon idealized mathematical models. The typical approach, first introduced into the literature by Nathanson in 1967 [14], is to create a "lumped" mass-spring model. Here, the elastic behavior of the system is represented by a linear spring while electrostatic forces are computed using a simple parallel plate capacitor approximation. This mass-spring model has persisted in the MEMS/NEMS literature and has been rediscovered and discussed by numerous authors, [6, 8, 10, 12, 13, 15, 17, 18]. Nevertheless, the mathematical analysis of this canonical model has remained primitive. Typically, authors have restricted their attention to steady-state solutions [12], relied upon numerical simulation for dynamical information [6], or have utilized perturbation methods to study approximate dynamics in some region of parameter space [18].

In this paper, we begin to remedy this situation by providing a rigorous analysis of the viscosity dominated time harmonically forced mass-spring MEMS/NEMS model. While this analysis does not capture the dynamics of every possible MEMS or NEMS device, it is relevant for the study of devices, such as micropumps [16], microgrippers [7], or nanotweezers [12], which operate in the viscosity dominated regime. In Section 2, for the convenience of the reader we provide a brief derivation of the model. In Section 3, we consider the situation where inertial forces are completely negligible. In this case, the model is reduced to a nonlinear first order non-autonomous ordinary differential equation. We study the existence of periodic solutions to this equation. We determine ranges of the dimensionless applied voltage and dimensionless forcing frequency for which such solutions exist. We show that outside of these ranges the model has no solution. This is the dynamic analog of the static instability well known to MEMS/NEMS researchers as the "pull-in" or "snap-down" instability. In this instability, when a constant applied voltage is increased beyond a certain critical voltage there is no longer a steady-state configuration of the device where mechanical members remain separate. Here in the dynamic situation, the device cannot be operated in an oscillatory mode if the mean applied voltage is too large or the forcing frequency is too small. In Section 4, we consider a situation where inertial forces are small, but non-negligible. In this case, the model becomes a nonlinear

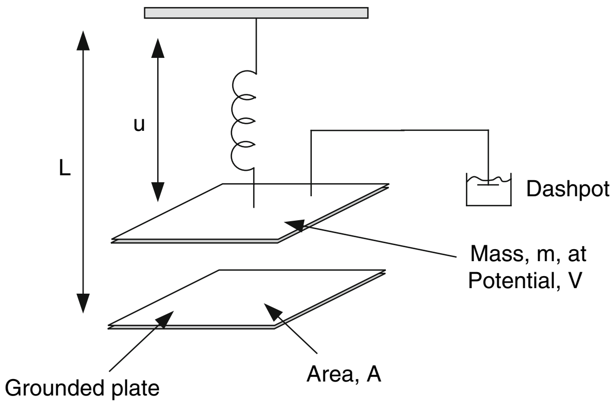

Figure 1. Sketch of the damped mass-spring system.

second order non-autonomous ordinary differential equation. Again we investigate periodic solutions of this equation and determine criteria necessary for such solutions to exist. Finally, in Section 5, we discuss the implications of our analysis for MEMS/NEMS device behavior.

2. FORMULATION OF THE CANONICAL MODEL

The system sketched in Fig. 1 represents a "lumped" approximation of a typical electrostatically actuated MEMS/NEMS device. The governing equation for this system is

$$m\frac{d^2x}{dt'^2} = F_s + F_d + F_e. \tag{2.1}$$

Here, x is the displacement of the top plate from the top wall and m is the top plate's mass. We assume that the bottom plate is held in place. The forces acting on our system are the spring force, F_s, a damping force represented by the dashpot in Fig. 1, F_d, and the electrostatic force, F_e, due to the applied voltage difference between the plates. We assume that the spring is a linear spring and follows Hooke's law

$$F_s = -k(x - l) \tag{2.2}$$

where l is the rest length of the spring and k is the spring constant. We assume that damping is linearly proportional to the velocity, that is

$$F_d = -a\frac{dx}{dt'} \tag{2.3}$$

and compute the electrostatic force by treating the plates in Fig. 1 as infinite parallel plates. This yields

$$F_e = \frac{1}{2}\frac{\varepsilon_0 A V^2}{(L-x)^2}\cos^2(\Omega t'). \tag{2.4}$$

Here, ε_0 is the permittivity of free space, A is the area of the plates, V is the average applied voltage, and Ω is the frequency at which the applied voltage is varied. Inserting equations (2.2), (2.3) and (2.4) into equation (2.1) yields

$$m\frac{d^2x}{dt'^2} + a\frac{dx}{dt'} + k(x-l) = \frac{1}{2}\frac{\varepsilon_0 A V^2}{(L-x)^2}\cos^2(\Omega t'). \tag{2.5}$$

We recast this equation in dimensionless form by introducing a dimensionless length scale

$$y = \frac{x-l}{L-l} \tag{2.6}$$

and dimensionless time scale

$$t = \frac{k}{a}t'. \tag{2.7}$$

Introducing equations (2.6) and (2.7) into equation (2.5) yields

$$\frac{1}{\alpha^2}\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = \frac{\lambda}{(1-y)^2}\cos^2(\gamma t), \tag{2.8}$$

where

$$\alpha^2 = \frac{a^2}{mk}, \quad \gamma = \frac{\Omega a}{k}, \quad \lambda = \frac{\varepsilon_0 A V^2}{2k(L-l)^3}.$$

The dimensionless parameter α may be interpreted as a damping coefficient which measures the relative strength of the viscous damping force as compared to the spring force. The dimensionless parameter λ measures the relative strength of electrostatic and elastic forces in our system. The dimensionless parameter γ is the ratio of damping and forcing time scales.

When damping effects dominate over inertial effects, we expect the parameter α to be large. If inertial effects are completely negligible, we send $\alpha \to \infty$ and study the reduced model

$$\frac{dy}{dt} + y = \frac{\lambda}{(1-y)^2}\cos^2(\gamma t). \tag{2.9}$$

This situation is studied in the next section. In Section 4, we return to the case where α is large, but inertial effects are not completely negligible in (2.5).

3. THE VISCOSITY DOMINATED CASE

In this section we study the $2\pi/\gamma$-periodic solutions of equation (2.9). We restrict our attention to solutions where $y<1$ as $y>1$ implies that the top plate in Fig. 1 has passed through the bottom plate. It is convenient to introduce the following scalings:

$$u(t) := [1 - y(t/3\omega)]^3, \quad \omega := 2\gamma/3,$$

and hence the problem reduces to finding solutions u, $0<u<1$, of the periodic boundary value problem

$$\omega u'(t) = u^{2/3}(t) - u(t) - \lambda \cos^2(t/2), \tag{3.1}$$

$$u(0) = u(2\pi). \tag{3.2}$$

Since any solution u of (3.1)–(3.2) satisfies $u(t)<1$ for $t \in \mathbb{R}$ (see Lemma 3.1 (i)), it suffices to study the positive solutions of (3.1)–(3.2). We note that, due to the non-differentiable term $u^{2/3}$ at $u=0$, the initial value problem of (3.1) at $t=\pi$ with $u(\pi)=0$ loses uniqueness on the side $t \geqslant \pi$ for sufficiently small $\lambda>0$, and results in the existence of non-negative solutions of (3.1)–(3.2) which become 0 at $t=\pi$ in $[0,2\pi]$. Although these solutions are not physically meaningful for (2.9), for completeness we include them in the following main result:

Theorem 3.1. *For any $\omega > \omega_0 := 2/3$, there exist a unique number $\lambda_0(\omega) \in (\frac{16}{6561\omega^2}, \frac{16}{243\omega^2})$ and a unique number $\lambda_b(\omega) \in (4/27, 8/27)$ with $\lim_{\omega\to\infty} \lambda_b(\omega) = 8/27$ such that*

(*i*) *If $\lambda_0 < \lambda < \lambda_b$, or $\lambda = \lambda_b$, or $\lambda > \lambda_b$, then (3.1)–(3.2) has exactly two, one, no positive solutions, respectively, and does not have any other non-negative solutions.*

(*ii*) *If $0 < \lambda \leqslant \lambda_0$, then (3.1)–(3.2) has exactly two nonnegative solutions: one is strictly positive, the other reaches zero only at $t=\pi$ in $[0,2\pi]$.*

Remark 3.1. $\omega_0 = 2/3$ is not optimal. Note that $\frac{16}{243\omega^2} < 4/27$ if $\omega > 2/3$. For each $\omega > \omega_0$, $\lambda_b(\omega)$ is the bifurcation value resulting from the saddle-node bifurcation of periodic solutions of (3.1). It can be shown that $\lambda_b(\omega)$ is a smooth function for $\omega \in (\omega_0, \infty)$. For each $\omega > \omega_0$, the existence of $\lambda_0(\omega)$ is due to the fact that the smaller of nonnegative solutions of (3.1)–(3.2) reaches 0 at $t=\pi$ if $\lambda < \frac{16}{6561\omega^2}$. It is important to note that this does not occur if the function $u^{3/2}$ is changed into u^σ with $2/3 < \sigma < 1$. Indeed, we can prove the following:

Theorem 3.2. *Let $2/3 < \sigma < 1$ and $\omega > 0$, and consider the equation*

$$\omega u' = u^\sigma - u - \lambda \cos^2(t/2). \tag{3.3}$$

There is a smooth and positive function $\lambda_\sigma(\omega)$ for $\omega \in (0, \infty)$ with $\lim_{\omega\to\infty} \lambda_\sigma(\omega) = (1-\sigma)(1/\sigma)^{\sigma/\sigma-1}$ such that (3.3) has precisely two, one, no 2π-periodic and positive solution(s) respectively if $0 < \lambda < \lambda_\sigma(\omega)$, or $\lambda = \lambda_\sigma(\omega)$, or $\lambda > \lambda_\sigma(\omega)$.

Below we prove Theorem 3.1 by a set of lemmas. In Lemma 3.1 we prove the nonexistence of nonnegative solutions of (3.1)–(3.2) if $\lambda > 8/27$. In Lemma 3.3, we show that (3.1)–(3.2) can have at most two nonnegative solutions. In Lemma 3.4, we establish sufficient conditions on λ and ω for the existence and nonexistence of nonnegative solutions that reach zero at $t = \pi$. In Lemmas 3.5 and 3.6, we show the existence of nonnegative solutions for (3.1)–(3.2) within certain ranges of values of λ and ω. Then using these lemmas and the properties of the Poincare map of (3.1) we complete the proof of Theorem 3.1. Roughly speaking, this Poincare map with any fixed λ and ω is concave down, and the corresponding graph moves down as the value of λ is increased, resulting in a saddle-node bifurcation at $\lambda = \lambda_b(\omega)$. See an example in Chapter one of [11]. After proving Theorem 3.1, we present Theorems 3.3 and 3.4 that describe the asymptotic behavior of the solutions of (3.1)–(3.2) as $\omega \to 0$ and $\omega \to \infty$, respectively.

Lemma 3.1. *Let $\lambda > 0$ and $\omega > 0$.*

(*i*) *If u is a nonnegative solution of (3.1)–(3.2), then $u(t) < 1$ for all $t \in [0, 2\pi]$; furthermore, if u is not strictly positive on $[0, 2\pi]$, then u reaches 0 only at $t = \pi$ in $[0, 2\pi]$.*

(*ii*) *If (a) $\lambda \geqslant \frac{8}{27}$ and $\omega > 0$, or (b) $\lambda > \frac{4}{27}$ and*

$$\omega \leqslant \frac{4}{3\sqrt{\lambda}} \left(\lambda - \frac{4}{27}\right)^{3/2}, \tag{3.4}$$

then (3.1)–(3.2) does not have any non-negative solution.

Proof. We first show (i). We note that if u is a solution of (3.1) with $u(t_0) = 1$ for some $t_0 \in [0, 2\pi]$, then $\omega u'(t_0) = -\lambda \cos^2(t_0/2)$. It follows that if $t_0 \neq \pi$, then $u'(t_0) < 0$ and, otherwise, if $t_0 = \pi$, then $u'(t_0) = 0$, and differentiating (3.1) two times yields $u''(t_0) = 0$ and $\omega u'''(t_0) = -\lambda < 0$, which implies that $u'(t) < 0$ for $|t - t_0| > 0$ sufficiently small. Therefore, either case yields that $u'(t) < 0$ for $|t - t_0| > 0$ sufficiently small. Consequently, any solution of (3.1) can assume the value one at most one time.

Now let u be a solution of (3.1)–(3.2). Since $\omega u'(t) \leqslant u^{2/3}(t) - u(t) < 0$ whenever $u(t) > 1$, it follows either that $u(t) < 1$ for all t or that $u(t_0) = 1$

for some $t_0 \in [0, 2\pi)$. However, the above assertion and the periodicity of u implies that the latter cannot happen. This shows the first part of (i).

The second part of (i) directly follows from the facts that u is periodic, non-negative, and $\omega u'(t) = -\lambda \cos^2(t/2) < 0$ whenever $u(t) = 0$ for $t \in [0, 2\pi] \setminus \{\pi\}$.

We use contradiction arguments to show (ii). Assume that (3.1)–(3.2) has a non-negative solution u. It follows from (i) that $0 \leqslant u(t) < 1$ for $t \in \mathbb{R}$ and so $u^{3/2}(t) - u(t) \leqslant 4/27$ for all $t \in \mathbb{R}$, where the equality holds only when $u(t) = 8/27$. Clearly, $u(\cdot) \not\equiv 8/27$. Hence, if $\lambda \geqslant 8/27$ and $\omega > 0$ as assumed in (ii) (a), then integrating (3.1) yields

$$0 = \omega[u(2\pi) - u(0)] = \int_0^{2\pi} [u^{2/3}(t) - u(t) - \lambda \cos^2(t/2)]\, dt < 2\pi \left(\frac{4}{27} - \frac{\lambda}{2}\right) \leqslant 0,$$

a contradiction. Here we have used the property of the function $F(u) := u^{2/3} - u$ for $u \in [0, 1]$; namely, F is increasing on $[0, 8/27]$ and decreasing on $[8/27, 1]$, $F(0) = F(1) = 0$ and $0 \leqslant F(u) \leqslant F(8/27) = 4/27$ for $u \in [0, 1]$. Assume that (ii) (b) holds. Since $\omega u'(t) \leqslant 4/27 - \lambda \cos^2(t/2)$ and $\sin t > t - t^3/6$ for $t > 0$ and $u(0) < 1$, it follows that, for $t > 0$,

$$\omega u(t) < \omega + \frac{4}{27} t - \frac{\lambda}{2} (t + \sin t) < \omega - \left(\lambda - \frac{4}{27}\right) t + \frac{1}{12} \lambda t^3. \tag{3.5}$$

Let $f(t)$ be the right-hand side of the last inequality. We find that f reaches its global minimum on $[0, \infty)$ at $\bar{t} = \frac{2}{\sqrt{\lambda}} \sqrt{\lambda - \frac{4}{27}}$ and $f(\bar{t}) = \omega - \frac{4}{3\sqrt{\lambda}} (\lambda - \frac{4}{27})^{3/2} \leqslant 0$ by virtue of (3.4). (3.5) then yields $\omega u(\bar{t}) < f(\bar{t}) \leqslant 0$, which contradicts the non-negativeness of $u(\cdot)$. This shows (ii), thereby completing the proof of Lemma 3.1. □

Lemma 3.2. *Let $\omega > 0$. Let $u(t, \alpha, \lambda)$ denote the solution of (3.1) with $u(0, \alpha, \lambda) = \alpha$, whose dependence on ω is suppressed. Assume that $u(t, \alpha, \lambda)$ exists and is positive for any $(t, \alpha, \lambda) \in [0, 2\pi] \times \mathcal{P}_{\alpha\lambda}$ where $\mathcal{P}_{\alpha\lambda}$ is a open set in the $\alpha\lambda$-plane. Let $h(\alpha, \lambda) = u(2\pi, \alpha, \lambda) - \alpha$. Then, h is smooth on $\mathcal{P}_{\alpha\lambda}$ and, for $(\alpha, \lambda) \in \mathcal{P}_{\alpha\lambda}$,*

$$h_{\alpha\alpha}(\alpha, \lambda) < 0, \qquad h_\lambda(\alpha, \lambda) < 0, \tag{3.6}$$

where $h_{\alpha\alpha} = \frac{\partial^2 h}{\partial \alpha^2}$ and $h_\lambda = \frac{\partial h}{\partial \lambda}$.

Proof. We denote the right-hand side of (3.1) by $f(t, u, \lambda)$ whose dependence on ω is suppressed. Let $p(\alpha, \lambda) := u(2\pi, \alpha, \lambda)$ denote the Poincare map of (3.1) for $(\alpha, \lambda) \in \mathcal{P}_{\alpha\lambda}$. By means of variational equations we get, for $(\alpha,\lambda)\in\mathcal{P}_{\alpha\lambda}$,

$$\begin{cases} p_\alpha(\alpha,\lambda) &= \exp\int_0^{2\pi}\frac{1}{\omega}f_u(s,u(s,\alpha,\lambda),\lambda)\,ds>0,\\ p_{\alpha\alpha}(\alpha,\lambda) &= p_\alpha(\alpha,\lambda)\cdot\int_0^{2\pi}\frac{1}{\omega}f_{uu}(s,u(s,\alpha,\lambda),\lambda)\cdot u_\alpha(s,\alpha,\lambda)\,ds,\\ p_\lambda(\alpha,\lambda) &= \int_0^{2\pi}\frac{1}{\omega}f_\lambda(s,u(s,\alpha,\lambda),\lambda)\cdot u_\alpha(s,\alpha,\lambda)\,ds,\end{cases}\tag{3.7}$$

where, for $s\in[0,2\pi]$, $u_\alpha(s,\alpha,\lambda)=\exp\int_0^s f_u(\tau,u(\tau,\alpha,\lambda),\lambda)\,d\tau$, and

$$\begin{cases} f_u(s,u(s,\alpha,\lambda),\lambda) &= \frac{2}{3}[u(s,\alpha,\lambda)]^{-1/3}-1,\\ f_{uu}(s,u(s,\alpha,\lambda),\lambda) &= -\frac{2}{9}[u(s,\alpha,\lambda)]^{-4/3}<0,\\ f_\lambda(s,u(s,\alpha,\lambda),\lambda) &= -\cos^2(s/2).\end{cases}\tag{3.8}$$

Then (3.6) follows immediately from (3.7) and (3.8). □

Lemma 3.3. *For any $\lambda>0$ and $\omega>0$, (3.1)–(3.2) can have at most two non-negative solutions; and if (3.1)–(3.2) does have two nonnegative solutions, then one of them must be strictly positive.*

Proof. We first show the second assertion of the lemma. Assume on a contrary that u_1 and u_2 are two solutions of (3.1)–(3.2) for some $\lambda>0$ and $\omega>0$ such that $0<u_2<u_1$ on $[0,2\pi]\setminus\{\pi\}$ and $u_1(\pi)=0$ and $u_2(\pi)=0$. Note that $\omega(u_1-u_2)'=a(t)(u_1-u_2)$ where $a(t)=(2/3)\int_0^1[\theta u_1(t)+(1-\theta)u_2(t)]^{-1/3}\,d\theta-1>0$ if $t\in[t_0,\pi)$ and t_0 is sufficiently close to π so that $0<u_i(t)<8/27$ for $t\in[t_0,\pi)$. It follows that u_1-u_2 is increasing on $[t_0,\pi]$ and thus $0=u_1(\pi)-u_2(\pi)>u_1(t_0)-u_2(t_0)>0$, a contradiction. This confirms the second assertion.

We next show the first part of the lemma. Assume on a contrary that (3.1)–(3.2) has more than two nonnegative solutions for some $\lambda>0$ and $\omega>0$. Let u_i $(i=1,2,3)$ be three of such solutions. From the above assertion, we can assume that $u_3<u_2<u_1$ for all $t\in[0,2\pi]$. We proceed the proof in two cases.

Case 1. Assume that $u_3(t)>0$ for $t\in[0,2\pi]$. Then we have $0<u_3<u_2<u_1$ on $[0,2\pi]$ and we can take $\delta>0$ sufficiently small such that $u(t,\alpha,\lambda)$ exists and is positive on $[0,2\pi]$ for any $\alpha\in I_\alpha:=(u_3(0)-\delta,u_1(0)+\delta)$. Thus, $h(\alpha):=h(\alpha,\lambda)$ is defined for $\alpha\in I_\alpha$ and, from (3.6), $h_{\alpha\alpha}(\alpha)<0$ for $\alpha\in I_\alpha$. This implies that h is concave down on I_α and therefore cannot have three zeros $u_i(0)$ $(i=1,2,3)$ in I_α.

Case 2. Assume that $u_3(\pi)=0$. Note that, letting $\Theta_i=u_1^{2/3}+u_1^{1/3}u_i^{1/3}+u_i^{2/3}$ for $i=2,3$, $\omega(u_1-u_i)'=\left\{\frac{1}{\Theta_i}[u_1^{1/3}+u_i^{1/3}]-1\right\}(u_1-u_i)$ and so, after a little algebra,

$$\omega\frac{d}{dt}\left(\ln\frac{u_1-u_2}{u_1-u_3}\right)=\frac{1}{\Theta_2\Theta_3}\left[u_1^{1/3}u_2^{1/3}+u_1^{1/3}u_3^{1/3}+u_2^{1/3}u_3^{1/3}\right]\left[u_3^{1/3}-u_2^{1/3}\right].$$

Since the right-hand side of the above equation is negative on $[0, 2\pi]$, it implies that $\ln[(u_1 - u_2)/(u_1 - u_3)]$ cannot be 2π-periodic, a contradiction. We note that this proof also works for case 1. This completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *Let* $0 < \lambda < 8/27$ *and* $\omega > 0$.

(*i*) *If* $\lambda < \frac{16}{6561\omega^2}$ *and* u *is a nonnegative solution of* (*3.1*)–(*3.2*) *with* $u < 8/27$, *then* $u(\pi) = 0$.

(*ii*) *If* $\lambda > \frac{16}{243\omega^2}$ *and* u *is a nonnegative solution of* (*3.1*)–(*3.2*), *then* $u(t) > 0$ *for all* $t \in \mathbb{R}$.

Proof. We first show (i). Assume that $u > 0$ on $[0, 2\pi]$. Then $0 < u < 8/27$ and $u^{2/3} - u = u^{2/3}(1 - u^{1/3}) > u^{2/3}/3$. It then follows from (3.1) and the inequality $\cos^2 t/2 \leqslant (t-\pi)^2/4$ that $\omega u' > \frac{1}{3}u^{2/3} - \frac{\lambda(t-\pi)^2}{4}$ on $[0, 2\pi]$. We note that the equation

$$\omega u' = \frac{1}{3}u^{2/3} - \frac{\lambda(t-\pi)^2}{4}$$

has a solution $\bar{u} = d(t-\pi)^3$ for some $d > (\frac{2}{27\omega})^3$. Hence, since $u(\pi) > 0 = \bar{u}(\pi)$, by a simple comparison argument we have $u(t) > d(t-\pi)^3$ for $t > \pi$, which contradicts the boundedness of u. Therefore, we must have $u(\pi) = 0$, showing (i).

We now show (ii). From Lemma 3.1 (i) it suffices to show that $u(\pi) > 0$. Assume on the contrary that $u(\pi) = 0$. Since $u(t) > 0$ for $t \in (\pi, 2\pi]$, it follows that $\omega u'(t) = u^{2/3}(t) - u(t) - \lambda\cos^2(t/2) < u^{2/3}(t)$ for $t \in (\pi, 2\pi]$. Integrating over $[t_0, t]$ where $\pi < t_0 < t \leqslant 2\pi$ and then sending $t_0 \to \pi^+$ yields $u(t) < (t-\pi)^3/(27\omega^3)$ and so $u^{2/3}(t) < (t-\pi)^2/(9\omega^2)$. Note that, for any given sufficiently small $\delta > 0$, there exists a $t_\delta \in (\pi, 2\pi)$ sufficiently close to π such that $\cos^2(t/2) = \sin^2[(t-\pi)/2] > (1-\delta)(t-\pi)^2/4$ for $t \in (\pi, t_\delta]$. It follows from the equation (3.1)

$$\omega u'(t) \leqslant u^{2/3}(t) - \lambda\cos^2(t/2) < \left[\frac{1}{9\omega^2} - \frac{(1-\delta)\lambda}{4}\right](t-\pi)^2. \tag{3.9}$$

Take an integration over $[\pi, t]$ for $t \in (\pi, t_\delta]$ to get

$$u(t) < \frac{1}{3\omega}\left[\frac{1}{9\omega^2} - \frac{(1-\delta)\lambda}{4}\right](t-\pi)^3 =: a_1(t-\pi)^3. \tag{3.10}$$

Hence, if $\omega \geqslant 2/(3\sqrt{(1-\delta)\lambda})$, then $a_1 \leqslant 0$ so that $u(t) < 0$ for $t \in [\pi, t_\delta)$ which is a contradiction. Otherwise, if $a_1 > 0$, we replace u in the right-hand side of the first inequality in (3.9) by $a_1(t-\pi)^3$ to get

$\omega u' < [a_1^{2/3} - (1-\delta)\lambda/4](t-\pi)^2$ for $t \in (\pi, t_\delta]$ and then an integration yields, for $t \in (\pi, t_\delta]$,

$$u(t) < \frac{1}{3\omega}\left[a_1^{2/3} - \frac{(1-\delta)\lambda}{4}\right](t-\pi)^3 =: a_2(t-\pi)^3.$$

Note that $a_2 < a_1$. If $a_2 \leqslant 0$, then the above inequality implies $u(t) < 0$ for $t \in (\pi, t_\delta]$, a contradiction again. Otherwise, we repeat the above argument to get, as long as $a_{k-1} > 0$ $(k \geqslant 1)$ and $t \in (\pi, t_\delta]$,

$$u(t) < a_k(t-\pi)^3, \quad a_k = \begin{cases} \frac{1}{27\omega^3} & \text{if } k = 0, \\ \frac{1}{3\omega}\left[a_{k-1}^{2/3} - (1-\delta)\lambda/4\right] & \text{if } k > 0. \end{cases} \tag{3.11}$$

We claim that there is a finite k such that $a_k \leqslant 0$. Assume that the claim is false. Then, $a_k > 0$ and (3.11) hold for all $k = 0, 1, \ldots$. Since $a_0 > a_1$, a simple induction shows that a_k is (strictly) monotonically decreasing with zero as their lower bound and thus $a_k \to \hat{a}$ for some $\hat{a} \in [0, a_0)$ such that $\hat{a}^{2/3} - 3\omega\hat{a} - (1-\delta)\lambda/4 = 0$. However, for any fixed $\omega > 4/(9\sqrt{3\lambda})$, we can choose a $\delta > 0$ sufficiently small at the beginning such that $\omega > 4/(9\sqrt{3(1-\delta)\lambda})$ and then it can be easily verified that the equation $a^{2/3} - 3\omega a - (1-\delta)\lambda/4 = 0$ does not have any nonnegative solution. This contradiction shows the above claim. Using this claim and (3.11) we get $u(t) < 0$ for $t \in (\pi, t_\delta]$, again a contradiction. Therefore, we conclude that $u(\pi) = 0$ does not hold, thereby completing the proof of (ii). □

Remark 3.2. Lemma 3.4 (i) was proved by Professor Stuart P. Hastings at the University of Pittsburgh. The authors thank him for providing this important result.

Lemma 3.5. *If* $0 < \lambda < 4/27$ *and* $\omega > 0$, *then* (*3.1*)–(*3.2*) *has two nonnegative solutions* u_1 *and* u_2 *that satisfy* $u_2 < \kappa_2 < \kappa_1 < u_1 < 1$ *on* $[0, 2\pi]$, *where* κ_1 *and* κ_2 *are the two roots of* $\kappa^{2/3} - \kappa - \lambda = 0$ *such that* $0 < \kappa_2 < 8/27 < \kappa_1 < 1$.

Proof. We first note that, since $0 < \lambda < 4/27$, the algebraic equation $\kappa^{2/3} - \kappa - \lambda = 0$ has exactly two roots $\kappa_i = \kappa_i(\lambda)$ $(i = 1, 2)$ in $[0, \infty)$ such that $0 < \kappa_2 < 8/27 < \kappa_1 < 1$ (see Fig. 2). We fix a $\lambda \in (0, 4/27)$ and an $\omega > 0$ and let $u(t, \alpha)$ denote the solution of the equation (3.1) with $u(0, \alpha) = \alpha$. Note that if $u(t, \alpha) = \kappa_1$ for some $t \in [0, 2\pi]$, then $\omega u'(t, \alpha) = \lambda(1 - \cos^2 t/2) = \lambda \sin^2 t/2 > 0$ except for $t = 0, 2\pi$. It follows that, for any $\kappa_1 \leqslant \alpha \leqslant 1$, $u(t, \alpha)$ exists for $t \in [0, 2\pi]$ and satisfies $\kappa_1 < u(t, \alpha) < 1$ for $t \in (0, 2\pi]$. This implies that the Poincare map $u(2\pi, \cdot)$ is well-defined on $[\kappa_1, 1]$ and maps $[\kappa_1, 1]$ into $(\kappa_1, 1)$. Thus, the intermediate value theorem yields that $u(2\pi, \cdot)$ has

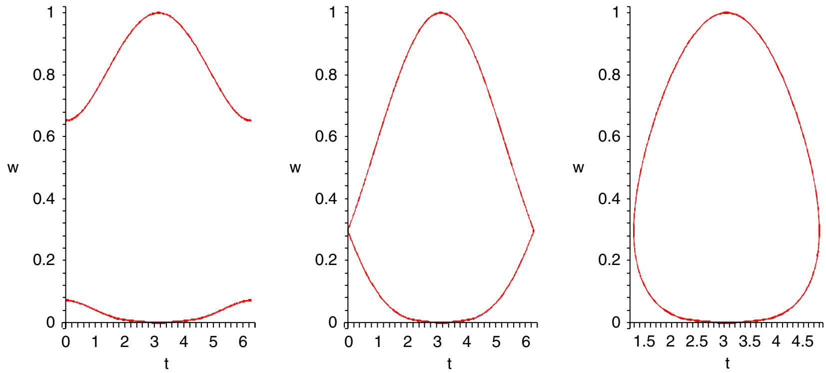

Figure 2. The curves of $u^{2/3} - u - \lambda\cos^2(t/2) = 0$ with $\lambda = 0.1$, $4/27$, 0.25, respectively.

at least one fixed point $\alpha_1 \in (\kappa_1, 1)$. Letting $u_1(t) := u(t, \alpha_1)$ gives a desired solution.

In order to show the existence of u_2, we consider the equation

$$\omega u' = u^{3/2} - u - \lambda\cos^2(t/2) - \eta, \tag{3.12}$$

where $0 < \eta < 4/27 - \lambda$. Let $u(t, \alpha, \eta)$ be the solution of (3.12) with $u(0, \alpha, \eta) = \alpha$. Let κ_η be the root of $\kappa^{2/3} - \kappa - \lambda - \eta = 0$ lying in $(0, 8/27)$. Then one verifies easily that, for each $\alpha \in [\eta^{3/2}, \kappa_\eta]$, $u(t, \alpha, \eta)$ exists for $t \in [-2\pi, 0]$ and satisfies $\eta^{3/2} < u(t, \alpha, \eta) < \kappa_\eta$ for $t \in [-2\pi, 0)$. Thus, the Poincare map $u(-2\pi, \cdot, \eta)$ has a fixed point $\alpha_\eta \in (\eta^{3/2}, \kappa_\eta)$ and $u(t, \eta) := u(t, \alpha_\eta, \eta)$ is a 2π-periodic and positive solution of (3.12).

We now apply the Arzela-Ascoli theorem to conclude that there is a sequence $\{\eta_n\}$ with $\eta_n \to 0$ as $n \to \infty$ such that the sequence $\{u(\cdot, \eta_n)\}_{n=1}^{\infty}$ converge to a 2π-periodic function, which is denoted by $u_2(\cdot)$. An easy exercise shows that u_2 is a solution of (3.1)–(3.2). Since $\kappa_{\eta_n} \to \kappa_2$ and $\eta^{3/2} < u(\cdot, \eta) < \kappa_\eta$, it follows that $0 \leqslant u_2 \leqslant \kappa_2$. An slope field analysis yields that $u_2(t) < \kappa_2$ for $t \in [0, 2\pi]$. This completes the proof of Lemma 3.5. $\square$

The next lemma concerns the existence of solutions of (3.1)–(3.2) as well but allows λ in a larger interval, namely, $0 < \lambda < 8/27$. The result of this lemma implies that $\lim_{\omega\to\infty} \lambda_b(\omega) = 8/27$.

Lemma 3.6. *Assume that* $0<\lambda<8/27$. *Let* $\gamma_1=\gamma_1(\lambda)$ *and* $\gamma_2=\gamma_2(\lambda)$ *be the solutions of the equation* $\gamma^{2/3}-\gamma-\lambda/2=0$. *Define*

$$\begin{aligned}
\delta &= \frac{1}{2}\min\left\{\gamma_2,\ \frac{8}{27}-\gamma_2,\ \gamma_1-\frac{8}{27},\ 1-\gamma_1\right\},\\
\gamma_{ij} &= \gamma_i+(-1)^j\delta, \qquad (i=1,2,\ j=1,2)\\
\Delta_{21} &= -\frac{27}{8\pi}\left(\frac{2}{3\gamma_{21}^{1/3}}-1\right)^{-1}\left(\gamma_{21}^{2/3}-\gamma_{21}-\frac{1}{2}\lambda\right),\\
\Delta_{22} &= \frac{27}{8\pi}\left(\frac{2}{3(\gamma_{22}-\frac{1}{2}\delta)^{1/3}}-1\right)^{-1}\left(\gamma_{22}^{2/3}-\gamma_{22}-\frac{1}{2}\lambda\right),\\
\Delta_{11} &= \frac{27}{8\pi}\left(\gamma_{11}^{2/3}-\gamma_{11}-\frac{1}{2}\lambda\right),\\
\Delta_{12} &= -\frac{27}{8\pi}\left(\gamma_{12}^{2/3}-\gamma_{12}-\frac{1}{2}\lambda\right),\\
\Delta &= \min\{\Delta_{11},\ \Delta_{12},\ \Delta_{21},\ \Delta_{22}\}.
\end{aligned}$$

If $\omega>\max\{16\pi/(27\delta),\ 1/\Delta\}$, *then* (*3.1*)–(*3.2*) *has exactly two positive solutions* u_1 *and* u_2 *such that*

$$|u_i(t)-\gamma_i|\leqslant\frac{3}{2}\delta \qquad \textit{for} \qquad t\in[0,2\pi]. \tag{3.13}$$

Proof. We first fix a λ and an ω that satisfy the assumptions of the lemma, and then let $\varepsilon=1/\omega$ and then write the equation (3.1) in the form

$$u'=\varepsilon[u^{2/3}-u-\lambda\cos^2(t/2)]. \tag{3.14}$$

We use $u(t,\alpha)$ to denote the solution of the equation (3.14) with $u(0,\alpha)=\alpha$.

We first show the existence of u_2. For any $\alpha\in[\gamma_{21},\gamma_{22}]=[\gamma_2-\delta,\gamma_2+\delta]$, we define $\hat{t}=\sup\{t\in(0,2\pi)\colon 0<u(\cdot,\alpha)<1 \text{ on } (0,t)\}$. Then for $t\in(0,\hat{t})$, $0<u^{2/3}(t,\alpha)-u(t,\alpha)\leqslant 4/27$ so that

$$\begin{aligned}
\left|\int_0^t (u^{2/3}(s,\alpha)-u(s,\alpha)-\lambda\cos^2(s/2))\,ds\right| &\leqslant \max\left\{\frac{8\pi}{27},\lambda\int_0^{2\pi}\cos^2(s/2)\,ds\right\}\\
&= \max\left\{\frac{8\pi}{27},\lambda\pi\right\}\leqslant\frac{8\pi}{27},
\end{aligned}$$

and

$$|u(t,\alpha)-\alpha|\leqslant\varepsilon\left|\int_0^t (u^{2/3}(s,\alpha)-u(s,\alpha)-\lambda\cos^2 s)\,ds\right|\leqslant\frac{8\pi}{27}\varepsilon, \tag{3.15}$$

which together with $\varepsilon < 27\delta/16\pi$ yields

$$0 < \gamma_2 - \frac{3}{2}\delta < \alpha - \frac{8\pi}{27}\varepsilon < u(t,\alpha) < \alpha + \frac{8\pi}{27}\varepsilon < \gamma_2 + \frac{3}{2}\delta < \frac{8}{27}. \qquad (3.16)$$

Consequently, $\hat{t} = 2\pi$ and (3.16) holds for $t \in [0, 2\pi]$.

Therefore, if $\alpha = \gamma_{21}$, using (3.16), the monotonicity of $F(u) = u^{2/3} - u$ for $u \in [0, 8/27]$, the inequality $(1+b)^{2/3} < 1 + 2b/3$ with $b > 0$ and the assumption on ω we obtain

$$\begin{aligned}
u(2\pi, \gamma_{21}) - \gamma_{21} &< \varepsilon \int_0^{2\pi} \left[\left(\gamma_{21} + \frac{8\pi}{27}\varepsilon\right)^{2/3} - \left(\gamma_{21} + \frac{8\pi}{27}\varepsilon\right) - \lambda \cos^2(s/2) \right] ds \\
&< \varepsilon \int_0^{2\pi} \left[\gamma_{21}^{2/3} + \frac{16\pi}{81\gamma_{21}^{1/3}}\varepsilon - \left(\gamma_{21} + \frac{8\pi}{27}\varepsilon\right) - \lambda \cos^2(s/2) \right] ds \\
&= 2\pi\varepsilon \left[\left(\gamma_{21}^{2/3} - \gamma_{21} - \frac{\lambda}{2}\right) + \frac{8\pi}{27}\left(\frac{2}{3}\gamma_{21}^{-1/3} - 1\right)\varepsilon \right] \\
&= \frac{16\pi^2\varepsilon}{27}\left(\frac{2}{3}\gamma_{21}^{-1/3} - 1\right)(-\Delta_{21} + \varepsilon) < 0. \qquad (3.17)
\end{aligned}$$

Similarly, using the inequality $(1-b)^{2/3} > 1 - 2b/[3(1-b)^{1/3}]$ for $0 < b < 1$ which is equivalent to $(1-b)^{1/3} < 1 - b/3$ for $0 < b < 1$ and $\varepsilon < 27\delta/(16\pi)$, we obtain

$$\begin{aligned}
u(2\pi, \gamma_{22}) - \gamma_{22} &> \varepsilon \int_0^{2\pi} \left[\left(\gamma_{22} - \frac{8\pi}{27}\varepsilon\right)^{2/3} - \left(\gamma_{22} - \frac{8\pi}{27}\varepsilon\right) - \lambda \cos^2(s/2) \right] ds \\
&> \varepsilon \int_0^{2\pi} \left[\gamma_{22}^{2/3} - \frac{16\pi\varepsilon}{81\gamma_{22}^{1/3}}\left(1 - \frac{8\pi\varepsilon}{27\gamma_{22}}\right)^{-1/3} - \left(\gamma_{22} - \frac{8\pi}{27}\varepsilon\right) \right. \\
&\quad \left. - \lambda \cos^2(s/2) \right] ds \\
&= 2\pi\varepsilon \left\{ \left[\gamma_{22}^{2/3} - \gamma_{22} - \frac{\lambda}{2} \right] - \frac{8\pi\varepsilon}{27}\left[\frac{2}{3}\left(\gamma_{22} - 8\pi\varepsilon/27\right)^{-1/3} - 1 \right] \right\} \\
&> 2\pi\varepsilon \left\{ \left[\gamma_{22}^{2/3} - \gamma_{22} - \frac{\lambda}{2} \right] - \frac{8\pi\varepsilon}{27}\left[\frac{2}{3}\left(\gamma_{22} - \delta/2\right)^{-1/3} - 1 \right] \right\} \\
&= \frac{16\pi^2\varepsilon}{27}\left[\frac{2}{3}\left(\gamma_{22} - \delta/2\right)^{-1/3} - 1 \right](\Delta_{22} - \varepsilon) > 0. \qquad (3.18)
\end{aligned}$$

Then, applying the intermediate value theorem to the continuous function $h(\alpha) := u(2\pi, \alpha) - \alpha$ for $\alpha \in [\gamma_{21}, \gamma_{22}]$ yields that there is a $\hat{\alpha} \in (\gamma_{21}, \gamma_{22})$

such that $h(\hat{\alpha})=0$ and thus $u_2(t):=u(t,\hat{\alpha})$ is a positive 2π-periodic solution of (3.14). Finally, using (3.15) and $\varepsilon<27\delta/16\pi$ we get, for $t\in[0,2\pi]$,

$$|u_2(t)-\gamma_2|\leqslant|\hat{\alpha}-\gamma_2|+|u_2(t)-\hat{\alpha}|\leqslant\delta+\frac{8\pi}{27}\varepsilon<\frac{3}{2}\delta, \tag{3.19}$$

which gives the estimate (3.13) for $i=2$.

To show the existence of u_1, we first note that for any $\alpha\in[\gamma_{12},\gamma_{12}]$, $u(t,\alpha)$ satisfies (3.15) as long as $0<u(t,\alpha)<1$. This yields that $u(\cdot,\alpha)$ exists on $[0,2\pi]$ and satisfies, for $t\in[0,2\pi]$,

$$\frac{8}{27}<\gamma_1-\frac{3\delta}{2}\leqslant\alpha-\frac{8\pi}{27}\varepsilon<u(t,\alpha)<\alpha+\frac{8\pi}{27}\varepsilon\leqslant\gamma_1+\frac{3\delta}{2}<1. \tag{3.20}$$

Then applying (3.20) with $\alpha=\gamma_{11}$ and $\alpha=\gamma_{12}$, respectively, the monotonicity of $F(u)$ for $u\in[8/27,1]$ and the assumption on ω we obtain

$$\begin{aligned}
u(2\pi,\gamma_{11})-\gamma_{11}&>\varepsilon\int_0^{2\pi}\left[\left(\gamma_{11}+\frac{8\pi}{27}\varepsilon\right)^{2/3}-\left(\gamma_{11}+\frac{8\pi}{27}\varepsilon\right)-\lambda\cos^2(s/2)\right]ds\\
&>\varepsilon\int_0^{2\pi}\left[\gamma_{11}^{2/3}-\left(\gamma_{11}+\frac{8\pi}{27}\varepsilon\right)-\lambda\cos^2(s/2)\right]ds\\
&=2\pi\varepsilon\left[\left(\gamma_{11}^{2/3}-\gamma_{11}-\frac{\lambda}{2}\right)-\frac{8\pi}{27}\varepsilon\right]\\
&=\frac{16\pi^2\varepsilon}{27}(\Delta_{11}-\varepsilon)>0,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
u(2\pi,\gamma_{12})-\gamma_{12}&<\varepsilon\int_0^{2\pi}\left[\left(\gamma_{12}-\frac{8\pi}{27}\varepsilon\right)^{2/3}-\left(\gamma_{12}-\frac{8\pi}{27}\varepsilon\right)-\lambda\cos^2(s/2)\right]ds\\
&<\varepsilon\int_0^{2\pi}\left[\gamma_{12}^{2/3}-\left(\gamma_{12}-\frac{8\pi}{27}\varepsilon\right)-\lambda\cos^2(s/2)\right]ds\\
&=2\pi\varepsilon\left[\left(\gamma_{12}^{2/3}-\gamma_{12}-\frac{\lambda}{2}\right)+\frac{8\pi}{27}\varepsilon\right]\\
&=\frac{16\pi^2\varepsilon}{27}(-\Delta_{12}+\varepsilon)<0.
\end{aligned} \tag{3.22}$$

Thus we conclude in the same manner as above that there exists a $\bar{\alpha}\in(\gamma_{11},\gamma_{12})$ such that $u_1(t):=u(t,\bar{\alpha})$ gives the desired solution u_1. The estimate in (3.13) for $i=1$ follows in the same way as in (3.19). □

Proof of Theorem 3.1. For $\omega>\omega_0=2/3$, we define the set

$$\Lambda(\omega)=\{\lambda>\lambda_0\colon\ (3.1)\text{–}(3.2)\text{ has two positive solutions}\}$$

which contains the interval $\left(\frac{16}{243\omega^2}, 4/27\right)$ from Lemmas 3.4 (ii) and 3.5, and therefore

$$\lambda_0(\omega) := \inf \Lambda(\omega), \qquad \lambda_b(\omega) = \sup \Lambda(\omega)$$

are well-defined and satisfy $\frac{16}{6561\omega^2} < \lambda_0(\omega) < \frac{16}{243\omega^2} < 4/27 < \lambda_b(\omega) < 8/27$ from Lemma 3.4 (i) and Lemma 3.1 (ii). From now on, we fix an $\omega > \omega_0$ and suppress the dependence of ω in $\lambda_0(\omega)$ and $\lambda_b(\omega)$. We want to show that $\Lambda(\omega) = (\lambda_0, \lambda_b)$. To this end, we first establish several claims.

Claim A. Assume that there is a $\bar{\lambda} > \lambda_0$ such that (a) for $\lambda = \bar{\lambda}$, (3.1)–(3.2) has a unique positive solution denoted by $\bar{u}$, and (b) there is an increasing sequence $\{\lambda_n\}$ with $\lim_{n\to\infty} \lambda_n = \bar{\lambda}$ such that (3.1)–(3.2) has two positive solutions $u_i(\cdot, \lambda_n)$ $(i = 1, 2)$ that satisfy $u_i(t, \lambda_n) \to \bar{u}(t)$ as $n \to \infty$ uniformly for $t \in [0, 2\pi]$. Then, there is a $\delta > 0$ such that (i) (3.1)–(3.2) has two positive solutions $u_i(\cdot, \lambda)$ $(i = 1, 2)$ for $\lambda \in (\bar{\lambda} - \delta, \bar{\lambda})$ that satisfy $u_i(t, \lambda) \to \bar{u}(t)$ as $\lambda \to \bar{\lambda}^-$ uniformly for $t \in [0, 2\pi]$; (ii) (3.1)–(3.2) does not have any positive solution that lies in a neighborhood of $\bar{u}(\cdot)$ for $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta)$. That is, a saddle-node bifurcation of periodic solutions of (3.1)–(3.2) occurs at $\lambda = \bar{\lambda}$ and $u = \bar{u}$.

Proof of Claim A. First, we take a $\delta_0 > 0$ sufficiently small such that, for $\alpha \in I_\alpha := (\bar{u}(0) - \delta_0, \bar{u}(0) + \delta_0)$ and $\lambda \in I_\lambda := (\bar{\lambda} - \delta_0, \bar{\lambda} + \delta_0)$, $u(t, \alpha, \lambda)$ is defined and positive for all $t \in [0, 2\pi]$. Applying Lemma 3.2 yields that $h(\alpha, \lambda) := u(2\pi, \alpha, \lambda) - \alpha$ is smooth and satisfies (3.6) on $\mathcal{P}_{\alpha\lambda} := I_\alpha \times I_\lambda$. It follows that $h(\cdot, \bar{\lambda})$ is concave down on I_α and has a unique zero $\bar{u}(0) \in I_\alpha$ by the assumption (a). Then the assumption (b) yields $\frac{\partial h}{\partial \alpha}(\bar{u}(0), \bar{\lambda}) = 0$. By Taylor's theorem we have, as $(\alpha, \lambda) \to (\bar{u}(0), \bar{\lambda})$,

$$h(\alpha, \lambda) = \left[h_\lambda(\bar{u}(0), \bar{\lambda}) + o(1)\right][\lambda - \bar{\lambda}] + \left[\frac{1}{2} h_{\alpha\alpha}(\bar{u}(0), \bar{\lambda}) + o(1)\right][\alpha - \bar{u}(0)]^2.$$

Since $h_\lambda(\bar{u}(0), \bar{\lambda}) \neq 0$ from (3.6), applying the implicit function theorem yields that there is a $\delta_1 \in (0, \delta_0)$ and a smooth function $\lambda = \lambda(\alpha)$ defined for $\alpha \in (\bar{u}(0) - \delta_1, \bar{u}(0) + \delta_1)$ such that $h(\alpha, \lambda(\alpha)) = 0$, $\lambda(\bar{u}(0)) = \bar{\lambda}$, and

$$\lambda(\alpha) = \bar{\lambda} - \frac{h_{\alpha\alpha}(\bar{u}(0), \bar{\lambda})}{2h_\lambda(\bar{u}(0), \bar{\lambda})}[1 + o(1)][\alpha - \bar{u}(0)]^2,$$

from which we get there there is a $\delta > 0$ such that, for $\lambda \in (\bar{\lambda} - \delta, \bar{\lambda}]$,

$$\alpha_\pm(\lambda) = \bar{u}(0) \pm \sqrt{\frac{2h_\lambda(\bar{u}(0), \bar{\lambda})}{h_{\alpha\alpha}(\bar{u}(0), \bar{\lambda})}}\,[1 + o(1)]\sqrt{\bar{\lambda} - \lambda}$$

which is smooth on $(\bar{\lambda} - \delta, \bar{\lambda})$. It is clear that $u_1(\cdot, \lambda) := u(\cdot, \alpha_+(\lambda), \lambda)$ and $u_2(\cdot, \lambda) := u(\cdot, \alpha_-(\lambda), \lambda)$ are two positive solutions of (3.1)–(3.2) for

$\lambda \in (\bar{\lambda} - \delta, \bar{\lambda})$. From Lemma 3.2, u_1 and u_2 are the only positive solutions of (3.1)–(3.2). The above formula also shows that (3.1)–(3.2) does not have any solutions near $\bar{u}$ for $\lambda \in (\bar{\lambda}, \bar{\lambda} + \delta)$. This completes the proof of Claim A.

Claim B. $\Lambda(\omega)$ is an open set in $(-\infty, \infty)$.

Proof of Claim B. Let $\hat{\lambda} \in \Lambda(\omega)$ and $u_i(\cdot, \hat{\lambda})$ $(i = 1, 2)$ be two positive solutions of (3.1)–(3.2). Then by the continuous dependence of solutions with initial conditions there is a $\delta_0 > 0$ such that $h(\alpha, \lambda) := u(2\pi, \alpha, \lambda) - \alpha$ is defined on $\mathcal{P}_{\alpha\lambda} := (u_2(0, \hat{\lambda}) - \delta_0, 1) \times (\hat{\lambda} - \delta_0, \hat{\lambda} + \delta_0)$ and $h(u_i(0, \hat{\lambda}), \hat{\lambda}) = 0$. Since $h(\cdot, \hat{\lambda})$ is concave down from (3.6), it follows that $h_\alpha(u_i(0, \hat{\lambda}), \hat{\lambda}) \neq 0$. Then applying the implicit function theorem yields that $h(\alpha, \lambda) = 0$ has two solutions $\alpha_i(\lambda) \in (u_2(0, \hat{\lambda}) - \delta_0, 1)$ for $\lambda \in (\hat{\lambda} - \delta, \hat{\lambda} + \delta)$ for some small $\delta \in (0, \delta_0)$ with $\alpha_i(\hat{\lambda}) = u_i(0, \hat{\lambda})$. Clearly, $u_i(\cdot, \lambda) := u_i(\cdot, \alpha_i(\lambda), \lambda)$ $(i = 1, 2)$ provide two positive solutions of (3.1)–(3.2) for $\lambda \in (\hat{\lambda} - \delta, \hat{\lambda} + \delta)$. Since $\hat{\lambda}$ is taken arbitrarily in $\Lambda(\omega)$, it follows that $\Lambda(\omega)$ is open.

Claim C. Assume that $[a, b] \in \Lambda(\omega)$. Let $u_2(\cdot, \lambda) < u_1(\cdot, \lambda)$ be the positive solutions of (3.1)–(3.2) for $\lambda \in [a, b]$. Then for each $t \in [0, 2\pi]$, $u_2(t, \cdot)$ is increasing and $u_1(t, \cdot)$ is decreasing with respect to $\lambda \in [a, b]$.

Proof of Claim C. Let $\alpha_i(\lambda) := u_i(0, \lambda)$ for $i = 1, 2$. By the continuity of solutions with initial data there is a $\delta > 0$ sufficiently small such that $u(t, \alpha, \lambda)$ is defined and positive for all $t \in [0, 2\pi]$ and $(\alpha, \lambda) \in \mathcal{P}_{\alpha\lambda} := \{(\alpha, \lambda) : \alpha \in (\alpha_2(\lambda) - \delta, 1), \lambda \in (a - \delta, b + \delta)\}$. Hence, $h(\alpha, \lambda) = u(2\pi, \alpha, \lambda) - \alpha$ is well-defined and smooth on $\mathcal{P}_{\alpha\lambda}$ and satisfies $h(\alpha_i(\lambda), \lambda) = 0$ for $\lambda \in (a - \delta, b + \delta)$. Applying Lemma 3.2 yields that $h(\cdot, \lambda)$ is concave down so that $h_\alpha(\alpha_1(\lambda), \lambda) < 0 < h_\alpha(\alpha_2(\lambda), \lambda)$. Then by the implicit differentiations we get

$$\alpha_1'(\lambda) = -\frac{h_\lambda(\alpha_1(\lambda), \lambda)}{h_\alpha(\alpha_1(\lambda), \lambda)} < 0, \quad \alpha_2'(\lambda) = -\frac{h_\lambda(\alpha_2(\lambda), \lambda)}{h_\alpha(\alpha_2(\lambda), \lambda)} > 0. \tag{3.23}$$

This implies that the assertions in Claim C hold for $t = 0$. To complete the proof, given arbitrary $\tilde{\lambda}, \hat{\lambda} \in (a - \delta, b + \delta)$ with $\tilde{\lambda} < \hat{\lambda}$, let $\tilde{u}_i(t) := u_i(t, \tilde{\lambda})$ and $\hat{u}_i(t) =: u_i(t, \hat{\lambda})$ for $i = 1, 2$. Since $\tilde{u}_2(0) = \alpha_2(\tilde{\lambda}) < \alpha_2(\hat{\lambda}_2) = \hat{u}_2(0)$, it follows that $\tilde{u}_2(2\pi) < \hat{u}_2(2\pi)$ and thus $\tilde{u}_2(t) < \hat{u}_2(t)$ for sufficiently small $t - 2\pi < 0$. Assume that there is a largest $t_0 \in (0, 2\pi)$ such that $\tilde{u}_2(t_0) - \hat{u}_2(t_0) = 0$. Using (3.1) we have $\omega(\tilde{u}_2'(t) - \hat{u}_2'(t)) = -(\tilde{\lambda} - \hat{\lambda})\cos^2(t/2)$, which together with $\tilde{u}_2'(t_0) - \hat{u}_2'(t_0) \leqslant 0$ implies that $t_0 = \pi$ and $\tilde{u}_2'(\pi) - \hat{u}_2'(\pi) = 0$. Then differentiating this equation two times successively yields that $\tilde{u}_2''(\pi) - \hat{u}_2''(\pi) = 0$ and $\tilde{u}_2'''(\pi) - u_2'''(\pi, \lambda_2) = -(\tilde{\lambda} - \hat{\lambda})/2 > 0$. Consequently, $\tilde{u}_2'(t) - \hat{u}_2'(t) > 0$ and $\tilde{u}_2(t) - \hat{u}_2(t) > 0$ for $t - \pi > 0$ sufficiently small. This contradicts the definition of t_0, whence $\tilde{u}(t) < \hat{u}(t)$ for all $t \in [0, 2\pi]$ as expected. In a similar fashion we show that $\tilde{u}_1(t) > \hat{u}_1(t)$ for $t \in [0, 2\pi]$. This completes the proof of Claim C.

Claim D. $u_1(\cdot, \lambda)$ exists for all $\lambda \in (0, \lambda_b)$.

Proof of Claim D. From the definition of λ_b we can apply Claim A with $\bar{\lambda} := \lambda_b$ to get that $u_1(\cdot, \lambda)$ exists for $\lambda \in (\lambda_b - \delta, \lambda_b)$ for some $\delta > 0$ with the property that $u_1(\cdot, \lambda)$ is decreasing with respect to λ and $\alpha_1'(\lambda) < 0$ from (3.23) where $\alpha_1(\lambda) = u_1(0, \lambda)$. We define

$$\begin{aligned}\bar{\lambda}_0 = \inf\{\bar{\lambda} \in (0, \lambda_b):&\\ &u_1(\cdot, \lambda) \text{ exists on } (\bar{\lambda}, \lambda_b) \text{ and is decreasing with respect to } \lambda\}.\end{aligned}$$

It follows that $0 \leqslant \bar{\lambda}_0 < \lambda_b$. Suppose that $\bar{\lambda}_0 > 0$. Then by Arzela-Ascoli theorem we have that $u_1(\cdot, \bar{\lambda}_0)$ exists and $u_1(\cdot, \bar{\lambda}_0) \geqslant u(\cdot, \lambda) > 0$ for $\lambda \in (\bar{\lambda}_0, \lambda_b)$ and $\lim_{\lambda \to \bar{\lambda}_0^+} u(t, \lambda) = u(t, \bar{\lambda}_0)$ uniformly for $t \in [0, 2\pi]$. Note that $h(\alpha, \lambda) := u(2\pi, \alpha, \lambda) - \alpha$ is defined for $\mathcal{P}_{\alpha\lambda} := (\alpha, \lambda) \in (\alpha(\bar{\lambda}_0) - \delta_0, 1) \times (\bar{\lambda}_0 - \delta_0, \bar{\lambda}_0 + \delta_0)$ for some sufficiently small $\delta_0 > 0$. Clearly $h(\alpha(\bar{\lambda}_0), \bar{\lambda}_0) = 0$. We must have $h_\alpha(\alpha(\bar{\lambda}_0), \bar{\lambda}_0) \neq 0$ for otherwise we would apply (3.6) to conclude that, for each λ in a right neighborhood of $\bar{\lambda}_0$, $h(\alpha, \lambda) = 0$ does not have any positive solution that is close to $u_1(\cdot, \bar{\lambda}_0)$ (i.e., a saddle-node bifurcation occurs for $h(\cdot, \lambda)$), a contradiction. Consequently, by the implicit function theorem we can extend $\alpha_1(\lambda)$ for $\lambda \in (\bar{\lambda}_0 - \delta, \bar{\lambda}_0 + \delta)$ such that $h(\alpha_1(\lambda), \lambda) = 0$ and $\alpha_1'(\lambda) < 0$ given in (3.23). This shows that $u_1(\cdot, \lambda)$ exists for all $(\bar{\lambda}_0 - \delta, \lambda_b)$ and is decreasing with respect to λ by the same argument used in the proof of Claim C. This contradicts the definition of $\bar{\lambda}_0$. Hence we must have $\bar{\lambda}_0 = 0$, showing Claim D.

We are ready to show that $\Lambda(\omega) = (\lambda_0, \lambda_b)$. First, the openness of $\Lambda(\omega)$ and Claim A (i) imply that there is a $\bar{\lambda}_0 \in [\lambda_0, \lambda_b)$ such that the open interval $(\bar{\lambda}_0, \lambda_b)$ is contained in $\Lambda(\omega)$ and $\bar{\lambda}_0 \notin \Lambda(\omega)$. We claim that $\bar{\lambda}_0 = \lambda_0$. Suppose that this is false. Since $\Lambda(\omega)$ is open, there exists at least another open interval $(\hat{\lambda}_0, \hat{\lambda}) \subset \Lambda(\omega)$ contained in $\Lambda(\omega)$ with $\hat{\lambda} \notin \Lambda(\omega)$ and $\hat{\lambda} > \bar{\lambda}_0$. This implies by the monotonicities of $u_1(\cdot, \lambda)$ and $u_2(\cdot, \lambda)$ from Claim C that $u_1(\cdot, \lambda)$ and $u_2(\cdot, \lambda)$ for $\lambda \in (\hat{\lambda}_0, \hat{\lambda})$ merges into $u_1(\cdot, \hat{\lambda})$ at $\lambda = \hat{\lambda}$ (whose existence follows from Claim D) so that $u_1(\cdot, \hat{\lambda})$ is the unique positive solution of (3.1)–(3.2). This is impossible since, by Claim A (ii) with $\bar{\lambda} := \hat{\lambda}$, (3.1)–(3.2) does not have any positive solution close to $u_1(\cdot, \hat{\lambda})$ for λ in a right neighborhood of $\hat{\lambda}$, contradicting that $u_1(\cdot, \lambda)$ exists for all $\lambda \in (0, \lambda_b)$ from Claim D. This contradiction shows that $\bar{\lambda}_0 = \lambda_0$, whence $\Lambda(\omega) = (\lambda_0, \lambda_b)$.

We now show that if $\lambda > \lambda_b$, then (3.1)–(3.2) does not have any nonnegative solution. Since $\lambda_b > \frac{16}{243\omega^2}$, it follows from Lemmas 3.4 (ii) that any nonnegative solution of (3.1)–(3.2) for $\lambda \geqslant \lambda_b$ is strictly positive. Therefore, it suffices to show that if $\lambda > \lambda_b$, then (3.1)–(3.2) does not have any positive solution. Assume that this is false. This implies that

$\tilde{\lambda} := \inf\{\lambda > \lambda_b : (3.1)\text{–}(3.2)$ has a unique positive solution$\}$ is well defined with $\tilde{\lambda} \geqslant \lambda_b$, and there are a sequence of $\{\lambda_n\}$ with $\lambda_n > \tilde{\lambda}$ and $\lambda_n \to \tilde{\lambda}$ as $n \to \infty$ and a sequence $\{u(\cdot, \lambda_n)\}$ of positive solutions of (3.1)–(3.2). Applying the Arzela-Ascoli theorem yields a convergent subsequence $\{u_{n_k}(\cdot, \lambda_{n_k})\}$ whose limit, denoted by $\tilde{u}$, gives the unique positive solution of (3.1)–(3.2) with $\lambda = \tilde{\lambda}$. Note that $h(\alpha, \lambda) := u(2\pi, \alpha, \lambda) - \alpha$ is well defined for $(\alpha, \lambda) \in (\tilde{u}(0) - \delta, \tilde{u}(0) + \delta) \times (\tilde{\lambda} - \delta, \tilde{\lambda} + \delta)$ for some sufficiently small $\delta > 0$. We must have $h_\alpha(\tilde{u}(0), \tilde{\lambda}) = 0$, for otherwise the implicit function theorem yields that $h(\alpha, \lambda) = 0$ has a solution for λ in a neighborhood of $\tilde{\lambda}$ so that (3.1)–(3.2) has a unique positive solution for $\lambda - \tilde{\lambda} < 0$ sufficiently small, contradicting the definition of $\tilde{\lambda}$. But then applying Lemma 3.2 yields that (3.1)–(3.2) cannot have solution lying in a neighborhood of $\tilde{u}$ for all $\lambda - \tilde{\lambda} > 0$ sufficiently small, again contradicting the definition of $\tilde{\lambda}$. This shows the above assertion.

Since $\lambda_b > \frac{16}{243\omega^2}$, it follows from Lemma 3.4 (ii) and the definition of λ_b that (3.1)–(3.2) has a unique positive position and no other nonnegative solution. Finally, Lemma 3.6 implies that $\lim_{\omega\to\infty} \lambda_b(\omega) = 8/27$. This completes the proof of Theorem 3.1 (i). Since $\lambda_0(\omega) < 4/27$ for $\omega > 2/3$, Theorem 3.1 (ii) follows directly from Lemma 3.5 and Lemma 3.3. This completes the proof of Theorem 3.1.

Theorem 3.3 below gives the leading order approximations for the solutions of (3.1)–(3.2) obtained in Theorem 3.1 as $\omega \to 0$, while Theorem 3.4 gives the first order approximations for these solutions as $\omega \to \infty$.

Theorem 3.3. *Assume that $0 < \lambda < \frac{4}{27}$ and $\omega > 0$. Let u_1 and u_2 be two nonnegative solutions of (3.1)–(3.2) obtained in Theorem 3.1 with $u_2 < u_1$. Let U_1 and U_2 be the 2π-periodic functions defined by the equation $U^{2/3} - U - \lambda\cos^2(t/2) = 0$ with $0 \leqslant U_2(t) < 8/27 < U_1(t) \leqslant 1$ for $t \in \mathbb{R}$. Then there are constants $M_i := M_i(\lambda) > 0$ $(i = 1, 2)$ independent of ω such that*

$$|u_i(t) - U_i(t)| \leqslant M_i\omega \quad \text{for } t \in [0, 2\pi]. \tag{3.24}$$

Proof. Let $\rho_1(t) = u_1(t) - U_1(t)$. Then

$$\omega\rho_1' = u_1^{2/3} - u_1 - \lambda\cos^2(t/2) - \omega U_1' = a_1(t)\rho_1 - \omega U_1',$$

where

$$U_1'(t) = -\frac{\lambda}{2}\sin t\left[\frac{2}{3}U_1^{-1/3}(t) - 1\right]^{-1}, \quad a_1(t) := \frac{2}{3}\int_0^1 [\theta u_1(t) + (1-\theta)U_1(t)]^{-1/3}\, d\theta - 1.$$

which are continuous for all t. Note that $\kappa_1 \leqslant U_1(t) \leqslant 1$, $\kappa_1 < u_1(t) < 1$ from Lemma 3.5 and $\kappa_1 > 8/27$ where κ_1 is defined in Lemma 3.5. It follows that

$a_1(t) \leqslant (2/3)\kappa_1^{-1/3} - 1 =: -\nu_1$. Applying the variation of constant formula yields

$$\rho_1(t) = e^{\int_0^t a_1(s)\,ds/\omega}\rho_1(0) - \int_0^t e^{\int_\eta^t a_1(s)\,ds/\omega} U_1'(\eta)\,d\eta,$$

and then using $|\rho_1(0)| = |u_1(0) - U_1(0)| < 1 - \kappa_1 < 1$ we have, for $t \geqslant \omega|\ln\omega|/\nu_1$,

$$\begin{aligned}|\rho_1(t)| &\leqslant |\rho_1(0)|e^{-\nu_1 t/\omega} + |U_1'|_0 \int_0^t e^{-\nu_1(t-\eta)/\omega}\,d\eta \\ &\leqslant \omega + \frac{|U_1'|_0}{\nu_1}\omega = \left(1 + \frac{|U_1'|_0}{\nu_1}\right)\omega\end{aligned}$$

where $|U_1'|_0 = \sup_{t\in\mathbb{R}} |U_1'(t)| = \max_{t\in[0,2\pi]} |U_1'(t)|$. Since ρ_1 is 2π-periodic, this implies that $|\rho_1(t)| \leqslant (1 + |U_1'|_0/\nu_1)\omega$ for all $t \in \mathbb{R}$. This shows (3.24) for $i = 1$ with $M_1 := 1 + |U_1'|_0/\nu_1$.

In remains show (3.24) with $i = 2$, In order to apply the above argument, we first need to verify the differentiability of U_2 at $t = \pi$ (since $U_2(\pi) = 0$). For $t \in [0, \pi) \cup (\pi, 2\pi]$, $U_2'(t)$ is given by the same formula as that of $U_1'(t)$ except in which $U_1(t)$ is replaced by $U_2(t)$. To obtain the differentiability of U_2 at π, we need the following asymptotic formulas as $t \to \pi$:

$$\begin{cases} U_2(t) = \lambda^{3/2}|\cos(t/2)|^3[1 + O(\cos(t/2))], \\ U_2'(t) = -\frac{3\lambda\sqrt{\lambda}}{4}\sin t|\cos(t/2)|[1 + O(\cos(t/2))]. \end{cases}$$

From these asymptotic formulas and the L'Hospital's rule, it follows

$$U_2'(\pi) = \lim_{t\to\pi} \frac{U_2(t) - U_2(\pi)}{t - \pi} = \lim_{t\to\pi} \frac{U_2'(t)}{1} = 0.$$

This also shows that U_2' is continuous at π so that U_2' is continuous for all t. We now let $\rho_2(t) := u_2(t) - U_2(t)$. In order to complete the proof, we distinguish two cases based on whether $u_2(\pi) > 0$ or $u_2(\pi) = 0$.

Case 1. Assume that $u_2(\pi) > 0$. Then $u_2(t) > 0$ for all $t \in \mathbb{R}$. Note that $\omega\rho_2'(t) = a_2(t)\rho_2(t) - \omega U_2'(t)$ for all $t \in \mathbb{R}$, where $a_2(t)$ is given by the same formula as that for $a_1(t)$ except in which $u_1(t)$ and $U_1(t)$ are replaced by $u_2(t)$ and $U_2(t)$ respectively. Since $0 < u_2(t) < \kappa_2$ and $0 \leqslant U_2(t) \leqslant \kappa_2$, it follows that $a_2(t)$ is continuous for $t \in \mathbb{R}$ and, furthermore,

$a_2(t) \geqslant (2/3)\kappa_2^{-1/3} - 1 := \nu_2 > 0$. Then using the variation of constant formula together with $|\rho_2(0)| < 1$ yields, for $t \leqslant -\omega|\ln\omega|/\nu_2$,

$$|\rho_2(t)| \leqslant |\rho_2(0)|e^{\nu_2 t/\omega} + |U_2'|_0 \int_t^0 e^{\nu_2(t-\eta)/\omega}\, d\eta$$
$$\leqslant \omega + \frac{|U_2'|_0}{\nu_2}\omega = \left(1 + \frac{|U_2'|_0}{\nu_2}\right)\omega.$$

Since ρ_2 is 2π-periodic, this implies that $|\rho_2(t)| \leqslant M_2\omega$ for all $t \in \mathbb{R}$ where $M_2 := 1 + |U_2'|_0/\nu_2$. This shows (3.24) for $i = 2$ in this case.

Case 2. Assume that $u_2(\pi) = 0$. Then $\rho_2(\pi) = 0$. It follows that, for any fixed $\omega > 0$, there is a $t_0 < \pi$ but sufficiently close to π such that $|\rho_2(t)| \leqslant \omega$ for $t \in [t_0, \pi]$. We now consider the equation $\omega\rho_2'(t) = a_2(t)\rho_2(t) - \omega U_2'(t)$ for $t \in (-\pi, t_0]$ where, since $u_2(t) > 0$, $a_2(t)$ is well-defined and $a_2(t) \geqslant \nu_2$. Again by the variation of constant formula and $|\rho_2(t_0)| \leqslant \omega$ we get, for $t \in (-\pi, t_0)$

$$|\rho_2(t)| \leqslant \rho_2(t_0)e^{\nu_2(t-t_0)/\omega} + |U_2'|_0 \int_t^{t_0} e^{\nu_2(t-\eta)/\omega}\, d\eta \leqslant \omega + \frac{|U_2'|_0}{\nu_2}\omega = M_2\omega,$$

where M_2 is the same as in Case 1. By letting $t \to -\pi^+$, we have $|\rho_2(-\pi)| \leqslant M_2\omega$. Since $M_2 > 1$, it follows that $|\rho_2(t)| \leqslant M_2\omega$ for all $t \in [-\pi, \pi]$. This shows (3.24) for $i = 2$ in Case 2, thereby completing the proof of Theorem 3.3. □

Theorem 3.4. *Assume that* $0 < \lambda < 8/27$. *Let* ω *be sufficiently large and* u_i $(i = 1, 2)$ *be solutions of* (3.1)–(3.2) *obtained in Theorem* 3.1 *with* $u_2 < u_1$. *Then there are constants* $M_i := M_i(\lambda) > 0$ *independent of* ω *such that*

$$\left|u_i(t) - \left(\gamma_i - \frac{\lambda \sin t}{2\omega}\right)\right| \leqslant \frac{M_i}{\omega^2} \quad \textit{for } t \in [0, 2\pi]. \tag{3.25}$$

Proof. Since the proofs of (3.25) for $i = 1, 2$ are almost the same, we only show (3.25) for $i = 1$. We let $\varepsilon := 1/\omega$ as we did in Lemma 3.6 so that u_1 satisfies (3.14). Hence, it suffices to show that, as $\varepsilon \to 0$,

$$u_1(t) = \gamma_1 - \frac{\lambda \sin t}{2}\varepsilon + O(\varepsilon^2) \qquad \text{uniformly for } t \in [0, 2\pi]. \tag{3.26}$$

Let $\alpha_1 := u_1(0)$. Since $|u_1(t) - \alpha_1| \leqslant (8\pi/27)\varepsilon$ for $t \in [0, 2\pi]$, it follows that $u_1(t) = \alpha_1 + O(\varepsilon)$ as $\varepsilon \to 0$ uniformly for $t \in [0, 2\pi]$. Then integrating (3.14) on $[0, 2\pi]$ yields, as $\varepsilon \to 0$, $0 = u_1(2\pi) - \alpha_1 = 2\pi(\alpha_1^{2/3} - \alpha_1 - \lambda/2)\varepsilon + O(\varepsilon^2)$, and so

$$\alpha_1^{2/3} - \alpha_1 - \frac{\lambda}{2} = O(\varepsilon). \tag{3.27}$$

Now, letting $e_1(t) = u_1(t) - \alpha_1 + \frac{\lambda \sin t}{2}\varepsilon$ yields

$$e_1'(t) = \varepsilon\left[u_1^{2/3}(t) - u_1(t) - \frac{\lambda}{2}\right] = \varepsilon\left[(\alpha_1^{2/3} - \alpha_1 - \frac{1}{2}\lambda) + O(\varepsilon)\right] \quad (\varepsilon \to 0)$$

uniformly for $t \in [0, 2\pi]$. It follows from (3.27) that $e_1'(t) = O(\varepsilon^2)$ uniformly for $t \in [0, 2\pi]$, which together with $e_1(0) = 0$ yields $e_1(t) = O(\varepsilon^2)$ uniformly for $t \in [0, 2\pi]$ and so

$$u_1(t) = \alpha_1 - \frac{\lambda \sin t}{2}\varepsilon + O(\varepsilon^2) \quad \text{uniformly for } t \in [0, 2\pi]. \tag{3.28}$$

To complete the proof, it remains to show that $\alpha_1 = \gamma_1 + O(\varepsilon^2)$ as $\varepsilon \to 0$. First, it follows from (3.28) and $\int_0^{2\pi} \sin t \, dt = 0$ that

$$\begin{aligned}
0 &= u_1(2\pi) - \alpha_1 \\
&= \varepsilon \int_0^{2\pi} \left[\left(\alpha_1 - \frac{\lambda \sin 2t}{4}\varepsilon + O(\varepsilon^2)\right)^{2/3} - \left(\alpha_1 - \frac{\lambda \sin 2t}{4}\varepsilon + O(\varepsilon^2)\right) - \lambda \cos^2 \frac{t}{2}\right] dt \\
&= \varepsilon\left[\left(\alpha_1^{2/3} - \alpha_1 - \frac{1}{2}\lambda\right)\pi + O(\varepsilon^2)\right],
\end{aligned}$$

and thus

$$\alpha_1^{2/3} - \alpha_1 - \frac{1}{2}\lambda = O(\varepsilon^2) \qquad (\varepsilon \to 0), \tag{3.29}$$

from which and the fact that $\alpha_1 \in (8/27, 1)$ we conclude that $\alpha_1 \to \gamma_1$ as $\varepsilon \to 0$ (for otherwise, the limit of left-hand side of (3.29) would not be zero as $\varepsilon \to 0$). Hence, $u_{11} := \alpha_1 - \gamma_1 = o(1)$ as $\varepsilon \to 0$ and so $\alpha_1^{2/3} = (\gamma_1 + u_{11})^{2/3} = \gamma_1 + 2u_{11}/3\gamma_1^{1/3} + O(u_{11}^2)$. Inserting this into (3.29) yields

$$\begin{aligned}
O(\varepsilon^2) &= \gamma_1^{2/3} - \gamma_1 - \frac{1}{2}\lambda + \left(\frac{2}{3\gamma_1^{1/3}} - 1\right)u_{11} + O(u_{11}^2) \\
&= \left(\frac{2}{3\gamma_1^{1/3}} - 1\right)u_{11} + O(u_{11}^2).
\end{aligned}$$

Since $2/3\gamma_1^{1/3} - 1 < 0$, we get $u_{11} = O(\varepsilon^2)$. Thus, $\alpha_1 = \gamma_1 + O(\varepsilon^2)$. Finally, substituting this into (3.28) leads to (3.26), thereby completing the proof of Theorem 3.4.

4. NON-NEGLIGIBLE INERTIAL EFFECTS

In this section we return to our basic model, equation (2.5), ignore the damping term and scale the time variable with respect to the forcing frequency. That is, we introduce the scalings

$$\frac{t}{2}=\Omega t', \quad v=\frac{L-x}{L-l}. \tag{4.1}$$

This yields

$$\varepsilon^2 v''=1-v-\frac{\lambda\cos^2(t/2)}{v^2}, \quad v>0, \tag{4.2}$$

where λ is as before and $\varepsilon:=\sqrt{m/(4k\Omega^2)}$ is the ratio of the forcing frequency to the natural frequency of the oscillator. Throughout the remainder of the paper, we use v_α to denote the solution of (4.2) satisfying

$$v(0)=\alpha>0, \quad v'(0)=0, \tag{4.3}$$

whose dependence on λ and ε is suppressed. It is easy to check that if $v'_\alpha(\pi)=0$, then v_α is symmetric about $t=\pi$ and so v_α is a 2π-periodic solution of (4.2). In this section, under the assumption that $0<\lambda<1/8$ and ε is sufficiently small, we prove there exist an order of $1/\varepsilon$ many such solutions for (4.2).

Before stating the main result of this section, we define, for $0<\lambda<4/27$, 2π-periodic functions $V_-=V_-(t)$ and $V_+=V_+(t)$ by the right-hand side of (4.2) such that $0<V_-(t)<2/3<V_+(t)<1$ for $t\in[0,\pi)$, $V_-(\pi)=0$ and $V_+(\pi)=1$. It is easy to see that V_- and V_+ are monotonically decreasing and increasing on $[0,\pi]$, respectively, and both are symmetric about $k\pi$ for any integer k. See their graphs in Fig. 3. Let b be the maximum value of V of the homoclinic orbit of $\ddot{V}=1-V-\lambda/V^2$. It follows that $b>V_+(0)>2/3$ and satisfies $b-b^2/2+\lambda/b=V_-(0)-V_-^2(0)/2+\lambda/V_-(0)$, which together with the equation $V_-^2(0)-V_-^3(0)=\lambda$ yields $b=2[1-V_-(0)]$ and thus $1<b<2$, or $b=1$, or $2/3<b<1$ if $0<\lambda<1/8$, or $\lambda=1/8$, or $1/8<\lambda<4/27$, respectively.

The main result of this section is as follows.

Theorem 4.1. *Let $0<\lambda<1/8$ and $\varepsilon>0$ be sufficiently small. Then there exists an integer $N:=N(\varepsilon,\lambda)>K_0/\varepsilon$ with $K_0>0$ independent of ε such that for each integer n with $2\leqslant n\leqslant N$, there exists a 2π-periodic solution $v_{n,\varepsilon}$ of (4.2) such that $v_{n,\varepsilon}$ has exactly $n+1$ critical points $0=:t_0<t_1<\dots,t_{n-1}<t_n=\pi$ in $[0,\pi]$, where $v_{n,\varepsilon}$ has local maxima and minima alternately.*

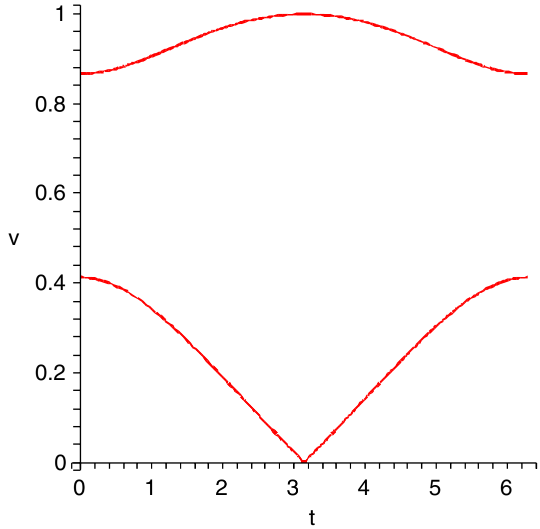

Figure 3. The top and bottom curves are the graphs of V_- and V_+ on $[0, 2\pi]$ respectively with $\lambda = 0.1$.

Remark 4.1. (i) Theorem 4.1 assures that the number of the 2π-periodic solutions of (4.2) goes to infinity as $\varepsilon \to 0$. We note that the assumption $0 < \lambda < 1/8$ is a technical assumption and is used only in Lemma 4.1 (ii) to guarantee that all critical points of v_α with $\alpha > 1$ are non-degenerate. (ii) The following asymptotic formulas for $v_{n,\varepsilon}$ as $\varepsilon \to 0$ can be obtained:

$$\begin{cases} t_i \to \pi, & 1 \leqslant i \leqslant n, \\ v_{n,\varepsilon}(0) \to b, & \\ v_{n,\varepsilon}(t_i) \to \pi & \text{if } i \geqslant 1 \text{ is odd}, \\ v_{n,\varepsilon}(t_i) \to 2 & \text{if } i \geqslant 2 \text{ is even}, \\ v_{n,\varepsilon}(t) = V_-(t) + O(\varepsilon^2) & \text{uniformly for } t \text{ on any compact subsets of } (0, \pi), \end{cases}$$

which assert that, as $\varepsilon \to 0$, $v_{n,\varepsilon}$ exhibits one spike near $t = (2k+1)\pi$ and the rest spikes near $t = 2k\pi$, and $v_{n,\varepsilon}(t)$ approaches $V_-(t)$ uniformly for t on any compact subsets of $(k\pi, (k+1)\pi)$ $(k = 0, \pm 1, \pm 2, \ldots)$. The proof of these formulas are similar to those in [1, 2] and is omitted.

The proof of Theorem 4.1 is based on several lemmas and shooting arguments that have been used in [1, 2, 9]. Lemma 4.1 shows that there is an $\alpha_0 \in (V_+(0), b)$ such that, for sufficiently small $\varepsilon > 0$, v_α has order of $1/\varepsilon$ many critical points in $(0, \pi)$; Lemma 4.2 shows that for any $\alpha \geqslant 1 + \sqrt{1 + 2\lambda}$, v_α does not have any critical point in $(0, \pi]$; Lemma 4.3 shows that no critical points of v_α in $[0,\pi]$ are degenerate with $\alpha > 1$. By virtue of these lemmas and the implicit function theorem we will be able to prove Theorem 4.1. Since the equation (4.2) is singular when $v=0$, we note that, before applying the implicit function theorem in the proof of Theorem 4.1, we have to prove a technical claim ensuring that the interested solutions are positive on $[0,\pi]$.

Lemma 4.1. *Let $0<\lambda<4/27$ and $\alpha_0 \in (V_+(0), b)$. Then there exist $K_0>0$ and $\varepsilon_0>0$ such that if $0<\varepsilon<\varepsilon_0$, then v_{α_0} has at least $N := N(\varepsilon,\lambda,\alpha_0) \geqslant K_0/\varepsilon$ critical points in $[0,\pi)$, which are local maximum and minimum points of v_{α_0} alternately.*

Since the proof of Lemma 4.1 is long and tedious, we leave it in the Appendix. See Fig. 4 for two solution curves of (4.2)–(4.3).

Lemma 4.2. *Assume that $0<\lambda<4/27$ and $\varepsilon>0$. Then for any $\alpha \geqslant 1+\sqrt{1+2\lambda}$, $v'_\alpha<0$ as long as $v_\alpha>0$ in $[0,\pi]$.*

Proof. We fix an $\alpha \geqslant 1+\sqrt{1+2\lambda}$ and let $v=v_\alpha$. Since $\alpha>1>V_+(0)$, it follows that $v'(t)<0$ for sufficiently small $t>0$. Assume that there is a $t_0 \in (0,\pi]$ such that $v(t)>0$ for $t\in[0,t_0]$, $v'(t)<0$ for $t\in[0,t_0)$ and $v'(t_0)=0$. Then, $v''(t_0)\geqslant 0$ which yields $V_-(t_0)\leqslant v(t_0)\leqslant V_+(t_0)\leqslant 1$. Multiply v' on both sides of (4.2) and integrate over $[0,t_0]$ to obtain

$$\begin{aligned}\alpha-\frac{1}{2}\alpha^2+\frac{\lambda}{\alpha} &= v(t_0)-\frac{1}{2}v^2(t_0)+\frac{\lambda\cos^2(t_0/2)}{v(t_0)}+\frac{\lambda}{2}\int_0^{t_0}\frac{\sin t}{v(t)}\,dt\\ &\geqslant v(t_0)-\frac{1}{2}v^2(t_0)\geqslant v(t_0)[1-V_+(t_0)/2]>0.\end{aligned} \tag{4.4}$$

But our assumption for α implies that $\alpha-\alpha^2/2+\lambda/\alpha < -(\alpha-1)^2/2+1/2+\lambda<0$. This yields a contradiction and thus the assertion of Lemma 4.3 holds. □

Lemma 4.3. *Assume that $0<\lambda<4/27$ and $\varepsilon>0$.*

(*i*) *If $t_0\in(0,\pi)$ is a minimum (resp. maximum) point of v_α and t_1 and t_2 are critical points of v_α in $[0,\pi]$ that are preceding and succeeding to t_0 respectively, then $v_\alpha(t_1)<v_\alpha(t_2)$.*

(*ii*) *Assume further that $0<\lambda<1/8$ and $\alpha\in(1,b)$. If $v'_\alpha(\tilde{t})=0$ and $\tilde{t}\in[0,\pi]$, then $v''_\alpha(\tilde{t})\neq 0$. Consequently, any critical point of v_α in $[0,\pi]$ is either a local maximum or minimum point.*

Proof. We first show (i). Assume that t_0 is a minimum point of v_α. Then, by the definition of t_1 and t_2, $v'_\alpha<0$ on (t_1,t_0) and $v'_\alpha>0$ on (t_0,t_2)

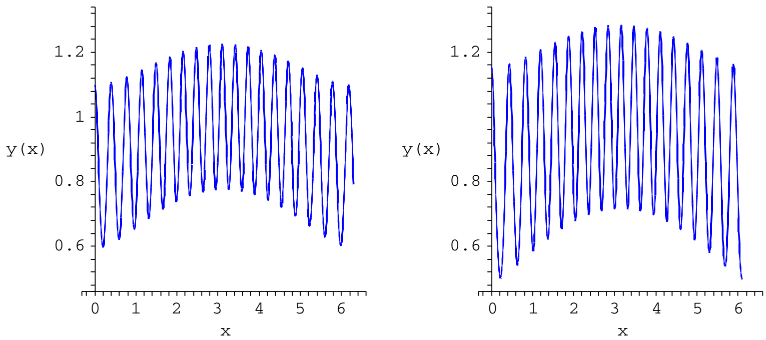

Figure 4. Two solution curves (4.2)–(4.3) with $\varepsilon = 0.05$, $\lambda = 0.1$, $\alpha = 1.1$ and 1.1552, respectively.

so that $v = v_\alpha(t)$ has an inverse on each of these two intervals, which are denoted by $t_- = t_-(v)$ and $t_+ = t_+(v)$, respectively. Multiplying (4.2) by v' and integrating over $[t_-(v), t_+(v)]$ for $v_\alpha(t_0) \leqslant v \leqslant \min\{v_\alpha(t_1), v_\alpha(t_2)\}$ we get

$$\frac{1}{2}\varepsilon^2(v_\alpha')^2(t_+(v)) - \frac{1}{2}\varepsilon^2(v_\alpha')^2(t_-(v)) = -\lambda \int_{v_\alpha(t_0)}^{v_\alpha(t_+(v))} \left(\cos^2\frac{t_+(\eta)}{2} - \cos^2\frac{t_-(\eta)}{2}\right)\frac{d\eta}{\eta^2}.$$

Since $\cos^2(t/2)$ is decreasing on $[0,\pi]$, it follows that the integral in the above formula is negative, which implies that $v_\alpha'(t_-(v))$ vanishes before $v_\alpha'(t_+(v))$ and thus $v_\alpha(t_1) < v_\alpha(t_2)$. The other case can be proved similarly. This shows (i).

We now show (ii). First we note that since $0<\lambda<1/8$, we have $b>1$ and thus the interval $(1,b)$ is not empty. Assume by a contradiction that $\tilde{t}\in[0,\pi]$ is the first degenerate critical point of v_α, i.e., $v_\alpha''(\tilde{t})=0$. Since $v_\alpha''(0) = -\lambda/\alpha^2 < 0$, we see that $\tilde{t} > 0$. Then, differentiating (4.2) yields $v_\alpha'''(\tilde{t}) = \lambda\sin\tilde{t}/2v_\alpha^2(\tilde{t}) > 0$ if $\tilde{t}\neq\pi$ and $v_\alpha'''(\tilde{t})=0$ and $v_\alpha''''(\tilde{t}) = \lambda\cos\tilde{t}/2v_\alpha^2(\tilde{t}) = -\lambda/2v_\alpha^2(\tilde{t}) < 0$ if $\tilde{t}=\pi$. Either case implies $v_\alpha'>0$ on $(t_0,\tilde{t})$ where $t_0\in(0,\tilde{t})$ is the preceding critical point of v_α whose existence follows from the fact that $v_\alpha'<0$ for sufficiently small $t>0$. Since we assume that $\tilde{t}$ is the first degenerate critical point of v_α, it follows that t_0 is a local minimum point of v_α, which together with Lemma 4.3 (i) yields $v_\alpha(\tilde{t}) > v_\alpha(0) = \alpha > 1 \geqslant V_+(\tilde{t})$ and thus $v_\alpha''(\tilde{t})<0$. This contradicts the assumption that $v_\alpha''(\tilde{t})=0$, whence proving (ii). □

Proof of Theorem 4.1. Since $0<\lambda<1/8$, it follows that $b>1$. Take an $\alpha_0\in(V_+(0),b)$ with $\alpha_0>1$ and $\bar{\alpha} := 1+\sqrt{1+2\lambda} > b$ (recall $b<2$). Fix an

$\varepsilon \in (0, \varepsilon_0)$ and an integer $2 \leqslant n \leqslant N(\varepsilon, \lambda, \alpha_0)$, where $\varepsilon_0 > 0$ and $N(\varepsilon, \lambda, \alpha_0)$ are given in Lemma 4.1 associated with α_0 chosen above. Define $\hat{\alpha} = \hat{\alpha}_{n,\varepsilon} := \sup A_{n,\varepsilon}$ where

$$A_{n,\varepsilon} = \{\alpha \in [\alpha_0, \bar{\alpha}] : v_\alpha \text{ has at least } n+1 \text{ distinct critical points in } [0, \pi)\}.$$

Lemma 4.1 implies $\alpha_0 \in A_{n,\varepsilon}$ and thus $A_{n,\varepsilon} \neq \emptyset$. Hence, $\hat{\alpha}$ is well-defined and there is a sequence $(\alpha_k)_{k=1}^{\infty}$ with $\alpha_k \in A_{n,\varepsilon}$ and $\alpha_k \to \hat{\alpha}$ as $k \to \infty$. Let $0 =: t_{k,0} < t_{k,1} < t_{k,2} < \cdots < t_{k,n} < \pi$ be the first $n+1$ critical points of v_{α_k} in $(0, \pi)$, which are all non-degenerate from Lemma 4.3 (ii) (since $\alpha_k \geqslant \alpha_0 > 1$). By the Bolzano-Weierstrass theorem we can assume, without loss of generality, that there exist $0 =: t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant \pi$ such that $t_{k,i} \to t_i$ as $k \to \infty$ for $i = 0, 1, \ldots, n$. Next, we claim:

Claim A. $v_{\hat{\alpha}} > 0$ on $[0, \pi]$.

We shall prove Claim A at the end of this proof. From Claim A, we know that $v_\alpha(t)$ is continuously differentiable for $t \in [0, \pi]$ and α in a neighborhood of $\hat{\alpha}$. It follows that $v'_{\hat{\alpha}}(t_i) = 0$ and, by virtue of Lemma 4.3 (ii) and $\hat{\alpha} > 1$, $v''_{\hat{\alpha}}(t_i) \neq 0$. We assert that $t_i \neq t_j$ for $0 \leqslant i < j \leqslant n$. Assume on a contrary that $t_i = t_{i+1}$ for some $0 \leqslant i \leqslant n-1$, then $t_{k,i} \to t_i$ and $t_{k,i+1} \to t_i$ as $k \to \infty$. However, since $v'_{\hat{\alpha}}(t_i) = 0$ and $v''_{\hat{\alpha}}(t_i) \neq 0$, the implicit function implies that there is a neighborhood of t_i such that v'_{α_k} has a unique zero in this neighborhood for all sufficiently large k. This yields a contradiction. Therefore, $0 = t_0 < t_1 < \cdots < t_n \leqslant \pi$.

We note that $A_{n,\varepsilon}$ is an open set in $\mathbb{R}$, which again follows from the non-degeneracy of critical points of v_α with $\alpha \in A_{n,\varepsilon}$ and the implicit function theorem. We thus concludes $\hat{\alpha} \notin A_{n,\varepsilon}$ and $t_n = \pi$. Letting $v_{n,\varepsilon} = v_{\hat{\alpha}}$ yields the desired solution as stated in Theorem 4.1. Therefore, to complete the proof of Theorem 4.1, it remains to prove Claim A.

Proof of Claim A. We start with some preliminary work. First, since $0 < \lambda < 4/27$ and $0 < V_- < 2/3$ on $(0, \pi)$, the implicit function theorem applies to the equation $V_-^2(t) - V_-^3(t) - \lambda \cos^2(t/2) = 0$ for $t \in (0, \pi)$ yielding that $V_-(t)$ is differentiable for $t \in (0, \pi)$ and

$$V'_-(t) = \frac{\lambda \sin t}{2V_-(t)[3V_-(t) - 2]} < 0$$

and thus V_- is monotonically decreasing on $[0, \pi]$. Furthermore, it is easy to derive that

$$V_-(t) = \sqrt{\lambda} \cos\frac{t}{2}\left(1 + O\left(\cos\frac{t}{2}\right)\right) \qquad \text{as } t \to \pi^-. \tag{4.5}$$

Second, we assert that if $\alpha > 1$ and v_α has its first minimum at $\bar{t} \in (0, \pi]$, then $v_\alpha(t)$ is defined for $t \in [0, \pi]$ with $v_\alpha(t) > V_-(\bar{t})$ for $t \in [0, \bar{t})$ and

$v_\alpha(t) > V_-(t)$ for all $t \in [\bar{t}, \pi]$. To see this, since $\bar{t}$ is a minimum point and $v''_\alpha(\bar{t}) \neq 0$, it follows that $v''_\alpha(\bar{t}) > 0$ and so $V_-(\bar{t}) < v_\alpha(\bar{t}) < V_+(\bar{t})$. Since the consecutive minimum values of v_α occurring in $[0, \pi]$ are increasing from Lemma 4.3 (i) and V_- is decreasing, the above assertion follows.

Now we are ready to prove Claim A. Below we use $v := v_{\hat{\alpha}}$ and $v_k := v_{\alpha_k}$ for simplicity. Assume that Claim A is false. Then, using the above assertion we conclude that there must exist a $\hat{t} \in (0, \pi]$ such that $v'(t) < 0$ for $t \in (0, \hat{t})$ and $v(t) \to 0$ as $t \to \hat{t}^-$. Then, for any $\eta > 0$, there is a $t_\eta \in (0, \hat{t})$ such that $0 < v(t) < \eta/2$ for $t \in [t_\eta, \hat{t})$. Hence, for sufficiently large k, $v_k(t_\eta) < \eta$ and $v'_k < 0$ on $(0, t_\eta]$. Recall that $t_{k,1}$ is the first minimum of v_k in $(0, \pi)$. It follows that $t_\eta < t_{k,1}$ and $v_k(t_{k,1}) < v_k(t_\eta) < \eta$. Since $n \geqslant 2$ (note that this is only place where we need $n \geqslant 2$), we see that $t_{k,2} \in (t_{k,1}, \pi)$ exists and $v_k(t_{k,2}) > v_k(0) = \alpha_k > 1$ by Lemma 4.3 (i). Let $\bar{t}_{k,1} \in (t_{k,1}, t_{k,2})$ such that $v_k(\bar{t}_{k,1}) = 1$. Then, $v'_k(t) > 0$ for $t \in (t_{k,1}, \bar{t}_{k,1}]$. This together with $v_k(t) > V_-(t)$ for $t \in [t_{k,1}, \pi]$ (from the above assertion), (4.2) and (4.5) yields that, for $t \in (t_{k,1}, \bar{t}_{k,1}]$,

$$\varepsilon |v''_k(t)| \leqslant 1 + \frac{\lambda \cos^2(t/2)}{V_-^2(t)} \leqslant 1 + \sup_{t \in [0,\pi)} \frac{\lambda \cos^2(t/2)}{V_-^2(t)} =: L,$$

and

$$v'_k(t) \leqslant \frac{1}{\varepsilon} \int_{t_{k,1}}^{t} |v''_k(t)|\, dt \leqslant \frac{L\pi}{\varepsilon} =: K,$$

and

$$1 - \eta \leqslant v_k(\bar{t}_{k,1}) - v_k(t_{k,1}) \leqslant K(\bar{t}_{k,1} - t_{k,1}). \tag{4.6}$$

Note that K does not depends on k and η.

Next we let $T_\eta \in (0, \pi)$ be defined by $V_-(T_\eta) = \eta$. Clearly, $T_\eta \to \pi$ as $\eta \to 0^+$. Let's fix an $\eta \in (0, 1/2)$ sufficiently small such that $\pi - T_\eta < 1/(2K)$. Then for sufficiently large k, since $V_-(t_{k,1}) < v_k(t_{k,1}) < v_k(t_\eta) < \eta = V_-(T_\eta)$ and V_- is decreasing on $[0, \pi]$, it follows that $T_\eta < t_{k,1}$ and hence, by (4.6), $1 - \eta \leqslant K(\bar{t}_{k,1} - t_{k,1}) < K(\pi - T_\eta) < 1/2$ so that $1 < 1/2 + \eta < 1$, a contradiction. This proves Claim A.

5. DISCUSSION

We began by presenting the canonical model of electrostatically actuated MEMS/NEMS devices. We studied the model in two limits. First, in Section 3, we considered the case where inertial terms were negligible relative to damping terms in this system. This led to the study of a non-autonomous nonlinear first-order differential equation. For this system, in the unforced case, it is well known that there are no steady-state solutions to the problem if $\lambda > 4/27$. That is, if the applied voltage is too large, the "pull-in" phenomena occurs. The principle result of Section 3 is the dynamic analog to the static pull-in phenomena. That is, we have shown that in the forced case, if the forcing frequency satisfies $\omega > 2/3$, then there are exactly one, two, and zero periodic solutions for $0 < \lambda \leqslant \lambda_0(\omega)$, $\lambda_0(\omega) < \lambda < \lambda_b(\omega)$, and $\lambda > \lambda_b(\omega)$ where $\frac{16}{6561\omega^2} < \lambda_0(\omega) < \frac{16}{243\omega^2} < 4/27 < \lambda_b(\omega) < 8/27$. Furthermore, $\lim_{\omega\to\infty} \lambda_b(\omega) = 8/27$, and there is a saddle-node bifurcation at $\lambda = \lambda_b(\omega)$. Physically, we have shown that if the system is forced with the correct frequency solutions can exist beyond the static pull-in voltage. Next, in Section 4, we considered the case where inertial terms were small but non-negligible and damping terms could be ignored. This led to the study of a non-autonomous nonlinear second order differential equation. A small parameter, ε, appeared in front of the inertial terms. The surprising result of this section was that as $\varepsilon \longrightarrow 0$ the number of periodic solutions tended to infinity. This indicated that in the presence of inertial terms, the canonical MEMS/NEMS model has much richer dynamics than in the viscosity dominated case.

6. APPENDIX

In this Appendix, we shall give the proof of Lemma 4.1 with some preliminary work prior to it.

We consider a family of autonomous equations with a parameter $\mu \in (0, 1]$

$$\ddot{V} = 1 - V - \frac{\lambda\mu}{V^2}, \qquad \text{where} \quad \ddot{V} = \frac{d^2 V}{d\tau^2}. \tag{6.1}$$

In the phase plane $(V, \dot{V})$, the equivalent first order system of (6.1) for each fixed μ has a unique homoclinic orbit that lies in the right half plane and connects the equilibrium point $(V_-(2\arccos\sqrt{\mu}), 0)$ of (6.1) at $\tau = \pm\infty$, whose interior is filled with closed orbits around the other equilibrium point $(V_+(2\arccos\sqrt{\mu}), 0)$ of (6.1) (see Fig. 5). Each of these closed orbits satisfies the equation

$$\dot{V}^2 = 2(V-\alpha) - (V^2 - \alpha^2) + 2\lambda\mu\left(\frac{1}{V} - \frac{1}{\alpha}\right),$$

for some $\alpha \in (V_+(2\arccos\sqrt{\mu}), V_-(2\arccos\sqrt{\mu}))$ which is the maximum of V and is symmetric about the V-axis with $\dot{V}$ having a unique minimum (resp. maximum) value at $V = V_+(2\arccos\sqrt{\mu})$ in the lower (resp. upper) half $(V, \dot{V})$-plane. We use $(V_{\alpha,\mu}(\tau), \dot{V}_{\alpha,\mu}(\tau))$ to denote a solution

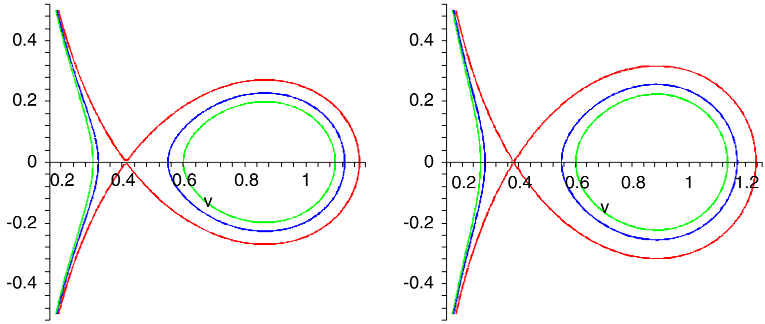

Figure 5. The phase plane diagrams of (6.1) with $\lambda = 0.1$, $\mu = 1$ and $\mu = 0.9$, respectively.

of (6.1) lying on such a orbit with $V(0) = \alpha$ and $\dot{V}(0) = 0$. Thus, $V_{\alpha,\mu}$ and $\dot{V}_{\alpha,\mu}$ are periodic functions of τ, whose least positive period is denoted by $2T_{\alpha,\mu}$; For any integer k, the graph of $V_{\alpha,\mu}$ in the (τ, V) plane is symmetric about $\tau = kT_{\alpha,\mu}$, and reaches its minimum value at $(2k+1)T_{\alpha,\mu}$ and its maximum value α at $2kT_{\alpha,\mu}$; $\dot{V}_{\alpha,\mu}$ is negative (resp. positive) having a unique minimum (resp. maximum) value in $(2kT_{\alpha,\mu}, (2k+1)T_{\alpha,\mu})$ (resp. $((2k+1)T_{\alpha,\mu}, (2k+2)T_{\alpha,\mu})$).

Proof of Lemma 4.1. Let $\tau := t/\varepsilon$, $\dot{v} = dv/d\tau$ and $\ddot{v} = d^2v/d\tau^2$. Then v_α satisfies

$$\ddot{v} = 1 - v - \frac{\lambda \cos^2(\varepsilon\tau/2)}{v^2}, \qquad v > 0, \tag{6.2}$$

$$v = \alpha, \quad \dot{v} = 0 \quad \text{at } \tau = 0. \tag{6.3}$$

We shall compare v_{α_0} with $V_{\alpha,\mu}$ of (6.1). To this end, we shall prove two claims.

Claim 1. Let $(V_0, \dot{V}_0) := (V_{\alpha_0,1}, \dot{V}_{\alpha_0,1})$ and $T_0 := T_{\alpha_0,1}$. There exist $\delta_0 > 0$ and $\delta_1 > 0$ sufficiently small and $\mu_0 \in (0, 1)$ sufficiently close to 1 such that for $\mu \in [\mu_0, 1]$ and $\alpha \in [\alpha_0 - \delta_0, \alpha_0 + \delta_0]$, the following (i) (ii) and (iii) hold:

(i) $V_{\alpha,\mu}(T_{\alpha,\mu}) \in [V_0(T_0) - \delta_1, V_0(T_0) + \delta_1]$;
(ii) $|1 - v - \lambda\mu/v^2| \geqslant \kappa$ for some $\kappa > 0$ as long as $v \in [\alpha_0 - 3\delta_0, \alpha_0 + 3\delta_0]$ or $v \in [V_0(T_0) - 3\delta_1, V_0(T_0) + 3\delta_1]$;
(iii) There is an $\eta > 0$ sufficiently small such that

$$\begin{cases} |\ddot{V}_{\alpha,\mu}(\tau)| \geqslant \kappa, & \text{if } \tau \in [0, \eta] \cup [T_{\alpha,\mu} - \eta, T_{\alpha,\mu} + \eta], \\ V_{\alpha,\mu}(\tau) \in [\alpha_0 - 2\delta_0, \alpha_0 + 2\delta_0], & \text{if } \tau \in [0, \eta] \cup [2T_{\alpha,\mu} - \eta, 2T_{\alpha,\mu}], \\ V_{\alpha,\mu}(\tau) \in [V_0(T_0) - 2\delta_1, V_0(T_0) + 2\delta_1], & \text{if } \tau \in [T_{\alpha,\mu} - \eta, T_{\alpha,\mu} + \eta], \\ |\dot{V}_{\alpha,\mu}(\tau)| \geqslant \kappa\eta, & \text{if } \tau \in [\eta, T_{\alpha,\mu} - \eta]. \end{cases}$$

Proof of Claim 1. (i) follows from the continuity of $V_{\alpha,\mu}$ with respect to (α, μ) at $(\alpha_0, 1)$ uniformly for τ in compact intervals and the continuity of $T_{\alpha,\mu}$ with respect to (α, μ). (ii) follows from the fact that $1 - V_0 - \lambda/V_0^2 \neq 0$ at $\tau = 0$ and $\tau = T_0$ and the continuity of the function $1 - v - \lambda/v^2$ at $v \neq 0$. Thus, (i) and (ii) are proved and δ_0, δ_1 and μ_0 are determined. To show (iii) we let $\delta_{01} = \min\{\delta_0, \delta_1\}$ and

$$\begin{aligned}
\tilde{T} &= \min\{T_{\alpha,\mu} : (\alpha, \mu) \in [\alpha_0 - \delta_0, \alpha_0 + \delta_0] \times [\mu_0, 1]\},\\
\bar{T} &= \max\{T_{\alpha,\mu} : (\alpha, \mu) \in [\alpha_0 - \delta_0, \alpha_0 + \delta_0] \times [\mu_0, 1]\},\\
M_0 &= \max\{|\dot{V}_{\alpha,\mu}(\tau)| : (\alpha, \mu) \in [\alpha_0 - \delta_0, \alpha_0 + \delta_0] \times [\mu_0, 1],\ \tau \in [0, T_{\alpha,\mu}]\},
\end{aligned}$$

and pick an $\eta > 0$ such that $\eta < \min\{\tilde{T}/2, \delta_{01}/M_0\}$. Let $(\alpha, \mu) \in [\alpha_0 - \delta_0, \alpha_0 + \delta_0] \times [\mu_0, 1]$. Then using the mean value theorem we have, for $\tau \in [0, \eta]$, $|V_{\alpha,\mu}(\tau) - \alpha| = |V_{\alpha,\mu}(\tau) - |V_{\alpha,\mu}(0)| \leqslant M_0\eta \leqslant \delta_0$ and so $\alpha_0 - 2\delta_0 \leqslant \alpha - \delta_0 \leqslant V_{\alpha,\mu}(\tau) \leqslant \alpha + \delta_0 \leqslant \alpha_0 + 2\delta_0$. Therefore, it follows from (ii) that $\ddot{V}_{\alpha,\mu}(\tau) = 1 - V_{\alpha,\mu}(\tau) - \lambda\mu/V_{\alpha,\mu}^2(\tau) \leqslant -\kappa$ for $\tau \in [0, \eta]$, and thus $\dot{V}_{\alpha,\mu}(\eta) = \int_0^\eta \ddot{V}_{\alpha,\mu}(s)\,ds \leqslant -\kappa\eta$. Similarly, for $\tau \in [T_{\alpha,\mu} - \eta, T_{\alpha,\mu} + \eta]$, $|V_{\alpha,\mu}(\tau) - V_{\alpha,\mu}(T_{\alpha,\mu})| \leqslant M_0\eta \leqslant \delta_1$ and so

$$V_0(T_0) - 2\delta_1 \leqslant V_{\alpha,\mu}(T_{\alpha,\mu}) - \delta_1 \leqslant V_{\alpha,\mu}(\tau) \leqslant V_{\alpha,\mu}(T_{\alpha,\mu}) + \delta_1 \leqslant V_0(T_0) + 2\delta_1.$$

Then, from (ii) we obtain $\ddot{V}_{\alpha,\mu}(\tau) \geqslant \kappa$ for $\tau \in [T_{\alpha,\mu} - \eta, T_{\alpha,\mu}]$ and so

$$\dot{V}_{\alpha,\mu}(T_{\alpha,\mu} - \eta) = \dot{V}_{\alpha,\mu}(T_{\alpha,\mu}) + \int_{T_{\alpha,\mu}}^{T_{\alpha,\mu}-\eta} \ddot{V}_{\alpha,\mu}(s)\,ds \leqslant -\kappa\eta.$$

Since $|\ddot{V}_{\alpha,\mu}| \geqslant \kappa$ for $\tau \in [0, \eta]$ or $\tau \in [T_{\alpha,\mu} - \eta, T_{\alpha,\mu} + \eta]$, we conclude that the unique minimum value of $\dot{V}_{\alpha,\mu}$ in $[0, T_{\alpha,\mu}]$ must occur in $(\eta, T_{\alpha,\mu} - \eta)$ and $|\dot{V}_{\alpha,\mu}(\tau)| \geqslant \min\{|\dot{V}_{\alpha,\mu}(\eta)|, |\dot{V}_{\alpha,\mu}(T_{\alpha,\mu} - \eta)|\} \geqslant \kappa\eta$ for $\tau \in [\eta, T_{\alpha,\mu} - \eta]$. This together with the symmetry of $V_{\alpha,\mu}$ and $\dot{V}_{\alpha,\mu}$ shows (iii).

Claim 2. Let μ_0 and δ_0 be given in Claim 1. There exist $M_1 > 0$ and $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and there is a $\sigma_n \geqslant 0$ such that $\dot{v}_{\alpha_0}(\sigma_n) = 0$, and

$$\mu_n := \cos^2(\varepsilon\sigma_n/2) > \mu_0, \qquad v_{\alpha_0}(\sigma_n) \in (\alpha_0 - \delta_0, \alpha_0 + \delta_0), \tag{6.4}$$

then there exist two successive critical points τ_{n+1} and σ_{n+1} of v_{α_0} after σ_n with $\sigma_n < \tau_{n+1} < \sigma_{n+1}$ such that

$$2\tilde{T} - \eta < \sigma_{n+1} - \sigma_n \leqslant 2\bar{T} + \eta, \quad |v_{\alpha_0}(\sigma_{n+1}) - v_{\alpha_0}(\sigma_n)| \leqslant M_1\varepsilon. \tag{6.5}$$

Proof of Claim 2. To show this claim, we need to compare $v_{\alpha_0}(\tau)$ with the solution $V_n(\tau) := V_{\alpha_n,\mu_n}(\tau-\sigma_n)$ of

$$\ddot V = 1 - V - \frac{\lambda\mu_n}{2}, \quad V(\sigma_n) = v_{\alpha_0}(\sigma_n) =: \alpha_n, \quad \dot V(\sigma_n) = 0.$$

By virtue of (6.4) we see that V_n has the properties (i)–(iii) listed in Claim 1 with a translation in τ. In order to show the existence of τ_{n+1} and σ_{n+1}, we first show

$$|v_{\alpha_0}(\tau) - V_n(\tau)| + |\dot v_{\alpha_0}(\tau) - \dot V_n(\tau)| \leqslant M_2\varepsilon, \quad \tau \in [\sigma_n, \sigma_n + 2T_n + \eta], \quad (6.6)$$

where $T_n := T_{\alpha_n,\mu_n}$, η is given in Claim 1, and $M_2 > 0$ is a constant depending only on δ_0, δ_1, μ_0 and η given in Claim 1. To this end, we let $w_n = v_{\alpha_0} - V_n$. Then w_n satisfies $w_n(\sigma_n) = \dot w_n(\sigma_n) = 0$ and

$$\begin{aligned}\ddot w_n &= -\left[1 - \frac{\lambda\,(2V_n + w_n)\cos^2(\varepsilon\tau/2)}{(w_n+V_n)^2 V_n^2}\right] w_n + \frac{\lambda\left[\cos^2(\varepsilon\tau_n/2) - \cos^2(\varepsilon\tau/2)\right]}{(w_n+V_n)^2} \\ &=: a_n(\tau)w_n + b_n(\tau).\end{aligned}$$

Define $T_n' = \sup\{\tau \in (\sigma_n, \sigma_n + 2T_n + \eta] : |w_n| + |\dot w_n| \leqslant \delta_{01}\ on\ [\sigma_n, \tau]\}$. Since $|\cos^2(\varepsilon\sigma_n/2) - \cos^2(\varepsilon\tau/2)| \leqslant \frac{1}{4}\varepsilon(\tau-\sigma_n)$ and $V_0(T_0) - \delta_1 \leqslant V_n(\tau) \leqslant \alpha_0 + \delta_0$, it follows that, for $\tau \in (\sigma_n, T_n']$,

$$|a_n(\tau)| \leqslant 1 + \frac{\lambda[2(\alpha_0+\delta_0)+\delta_{01}]}{[V_0(T_0)-2\delta_1]^2\,[V_0(T_0)-\delta_1]^2}, \quad |b(\tau)| \leqslant \frac{\lambda}{[V_0(T_0)-2\delta_1]^2}.$$

Letting M_3 be the sum of the right-hand sides of the above inequalities yields $|a_n(\tau)| \leqslant M_3$ and $|b_n(\tau)| \leqslant M_3$ for $\tau[\sigma_n, T_n']$. Using the equivalent integral equations for w_n and $\dot w_n$ we get, for $\tau \in (\sigma_n, T_n']$,

$$\begin{aligned}|w_n(\tau)| + |\dot w_n(\tau)| &\leqslant \int_{\sigma_n}^{\tau} (|a_n(s)|+1)(|w_n(s)| + |\dot w_n(s)|)\,ds + \int_{\sigma}^{\tau} |b_n(s)|\,ds \\ &\leqslant (M_3+1)\int_{\sigma_n}^{\tau} (|w_n(s)| + |\dot w_n(s)|)\,ds + M_3\bar T\varepsilon,\end{aligned}$$

and then the Gronwall's inequality and the estimate $\tau - \sigma_n \leqslant 2\bar T + \eta$ yields

$$|w_0(\tau)| + |\dot w_0(\tau)| \leqslant M_3\bar T e^{(M_3+1)(2\bar T+\eta)}\varepsilon =: M_2\varepsilon < M_2\varepsilon_0. \quad (6.7)$$

Hence if $\varepsilon_0 < \delta_{01}/M_2$, then $|w_0| + |\dot w_0| < \delta_{01}$ which yields by definition of T_n' that $T_n' = \sigma_n + 2T_n + \eta$ which in turn says that (6.7) holds on $[\sigma_n, \sigma_n + 2T_n + \eta]$. This yields (6.6) exactly.

We next use (6.6) to show the existence of τ_{n+1} and σ_{n+1}. It follows from (6.6) and Claim 1 (iii) that, for $\tau \in (\sigma_n, \sigma_n + \eta)$, $|v_{\alpha_0}(\tau) - \alpha_0| \leqslant$

$M_2\varepsilon + |V_n(\tau) - \alpha_0| \leqslant \delta_0 + 2\delta_0 = 3\delta_0$ and thus, from Claim 1 (ii), $\ddot{v}_{\alpha_0}(\tau) \leqslant -\kappa$ and thus $\dot{v}_{\alpha_0}(\tau) < 0$. For $\tau \in (\sigma_n + \eta, \sigma_n + T_n - \eta)$, we have $\dot{v}_{\alpha_0}(\tau) \leqslant \dot{V}_n(\tau) + M_2\varepsilon \leqslant -\kappa\eta + M_2\varepsilon_0 \leqslant -\kappa\eta/2$ provided $\varepsilon_0 \leqslant \kappa\eta/2M_2$. For $\tau \in (\sigma_n + T_n - \eta, \sigma_n + T_n + \eta)$, we have $|v_{\alpha_0}(\tau) - V_0(T_0)| \leqslant |v_{\alpha_0}(\tau) - V_n(\tau)| + |V_n(\tau) - V_0(T_0)| \leqslant M_2\varepsilon + 2\delta_1 < 3\delta_1$ and so $\ddot{v}_{\alpha_0}(\tau) \geqslant \kappa$ from Claim 1 (ii). Hence, $|\dot{v}_{\alpha_0}(\sigma_n + T_n \pm \eta)| \geqslant \kappa\eta - M_2\varepsilon \geqslant \kappa\eta/2$. Therefore, there exists a unique $\tau_{n+1} \in (\sigma_n + T_n - \eta, \sigma_n + T_n + \eta)$ such that $\dot{v}_{\alpha_0}(\tau_{n+1}) = 0$. Similarly, we have for $\tau \in (\sigma_n + T_n + \eta, \sigma_n + 2T_n - \eta)$, $|\dot{v}_{\alpha_0}(\tau)| \geqslant \kappa\eta - M_2\varepsilon \geqslant \kappa\eta/2$; for $\tau \in [\sigma_n + 2T_n - \eta, \sigma_n + 2T_n + \eta]$, $|v_{\alpha_0}(\tau) - \alpha_0| \leqslant |V_n(\tau) - \alpha_0| + M_2\varepsilon \leqslant 2\delta_0 + M_2\varepsilon \leqslant 3\delta_0$, and so $\ddot{v}_{\alpha_0}(\tau) \leqslant -\kappa$. Since $\dot{v}_{\alpha_0}(\sigma_n + 2T_n + \eta) \leqslant \dot{V}_n(\eta) + M_2\varepsilon \leqslant -\kappa\eta + M_2\varepsilon \leqslant -\kappa\eta/2$, and $\dot{v}_{\alpha_0}(\sigma_n + 2T_n - \eta) \geqslant -\dot{V}_n(\eta) - M_2\varepsilon \geqslant \kappa\eta - M_2\varepsilon \geqslant \kappa\eta/2$, it follows that there is a unique $\sigma_{n+1} \in (\sigma_n + 2T_n - \eta, \sigma_n + 2T_n + \eta)$ such that $\dot{v}_{\alpha_0}(\sigma_{n+1}) = 0$.

We now show (6.5). On one hand, we have $|v_{\alpha_0}(\sigma_{n+1}) - v_{\alpha_0}(\sigma_n)| = |v_{\alpha_0}(\sigma_{n+1}) - V_n(\sigma_n)| = |v_{\alpha_0}(\sigma_{n+1}) - V_n(\sigma_n + 2T_n)| \leqslant |v_{\alpha_0}(\sigma_{n+1}) - V_n(\sigma_{n+1})| + |V_n(\sigma_{n+1}) - V_n(\sigma_n + 2T_n)| \leqslant M_2\varepsilon + M_0|\sigma_{n+1} - (\sigma_n + 2T_n)|$. On the other hand, we have

$$
\begin{aligned}
0 = \dot{v}_{\alpha_0}(\sigma_{n+1}) - \dot{V}_n(\sigma_n + 2T_n) &= [\dot{v}_{\alpha_0}(\sigma_{n+1}) - \dot{V}_n(\sigma_{n+1})] \\
&\quad + [\dot{V}_n(\sigma_{n+1}) - \dot{V}_n(\sigma_n + 2T_n)]
\end{aligned}
$$

and so $M_2\varepsilon \geqslant |\dot{V}_n(\sigma_{n+1}) - \dot{V}_n(\sigma_n + 2T_n)| \geqslant \kappa|\sigma_{n+1} - (\sigma_n + 2T_n)|$ and so $|\sigma_{n+1} - (\sigma_n + 2T_n)| \leqslant M_2\varepsilon/\kappa$. Therefore, we have $|v_{\alpha_0}(\sigma_{n+1}) - v_{\alpha_0}(\sigma_n)| \leqslant M_2(1 + M_0/k)\varepsilon =: M_1\varepsilon$ and $2\tilde{T} - \eta \leqslant 2T_n - M_2\varepsilon_0/\kappa \leqslant \sigma_{n+1} - \sigma_n \leqslant 2T_n + M_2\varepsilon_0/\kappa \leqslant 2\bar{T} + \eta$ provided that $M_2\varepsilon_0/k \leqslant \eta$. This shows (6.5) as well as Claim 2.

Now we start proving Lemma 4.1. Given an arbitrary $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0 > 0$ is determined in Claim 2. Since $\sigma = 0$ and $v_{\alpha_0}(\sigma_0) = \alpha_0$, it follows that (6.4) is satisfied with $n = 0$. Then Claim 2 and the mathematical induction yields that there exist $0 =: \sigma_0 < \tau_1 < \sigma_1 < \tau_2 < \cdots < \tau_N < \sigma_N$ with

$$
N = N(\lambda, \varepsilon) := \sup\left\{ n \geqslant 1 :\ \cos^2(\varepsilon\sigma_n/2) > \mu_0, \qquad v_{\alpha_0}(\sigma_n) \in (\alpha_0 - \delta_0, \alpha_0 + \delta_0) \right\}.
$$

such that, for $n = 0, 1, \ldots, N$, v_{α_0} has local maxima and minima at σ_n and τ_n respectively and (6.5) holds. Note that from (6.5) and the definition of N we have $\sigma_{n+1} - \sigma_n \geqslant 2\tilde{T} - \eta$ and $\sigma_n \leqslant \bar{\delta}/\varepsilon$ for $0 \leqslant n < N$ where $\bar{\delta} := 2\arccos\sqrt{\mu_0}$. It follows that $N < \infty$. Therefore, we have (6.4) for $n = N$. Then Claim 2 implies that τ_{N+1} and σ_{N+1} exist and satisfy (6.5). On the other hand, by the definition of N, we must have either $\cos^2(\varepsilon\sigma_{N+1}/2) \leqslant \mu_0$ or $|v_{\alpha_0}(\sigma_{N+1}) - \alpha_0| \geqslant \delta_0$. If the former occurs, then, from (6.5),

$$
\frac{\bar{\delta}}{\varepsilon} \leqslant \sigma_{N+1} = (\sigma_{N+1} - \sigma_N) + \cdots + (\sigma_1 - \sigma_0) \leqslant (N+1)(2\bar{T} + \eta);
$$

while if the latter occurs, then, by 6.5) and the triangle inequality,

$$\delta_0 \leqslant |v_{\alpha_0}(\sigma_{N+1}) - v_{\alpha_0}(\sigma_N)| + \cdots + |v_{\alpha_0}(\sigma_1) - \alpha_0| \leqslant (N+1)M_1\varepsilon.$$

Thus, $N \geqslant K_0 := \min\{\delta_0/(M_1\varepsilon), \bar{\delta}/[(2\bar{T}+\eta)\varepsilon]\} - 1$. Finally, letting $t_{2k} = \varepsilon\sigma_n$ and $t_{2k+1} = \varepsilon\tau_n$ for nonnegative integer k with $2k+2 \leqslant N$ completes the proof of Lemma 4.1.

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