QUANTUM MECHANICS

Work 3 of the following 4 problems.

1. Consider the quantum mechanical bound states of the Earth-Sun system that satisfy

$$\left[\frac{-\hbar^2}{2M_E}\nabla^2 - \frac{GM_E M_S}{r}\right]\psi = E_n\psi,$$

where M_E and M_S are the masses of earth and sun, G is the universal gravitational constant.

A. From the analogy to the hydrogen atom, what plays the role of the fine structure constant α and the Bohr radius a_0? (For hydrogen atom $\alpha = e^2/\hbar c$).

B. For nearly circular orbits the angular momentum quantum number ℓ = n -1 and the wave function is of the form $\psi = Nr^{n-1}\exp(-r/na_0)Y_{n-1,m}(\theta,\phi)$. Use this to calculate the expectation value <r>.

C. Express the energy $E_n = -\frac{M_E c^2 \alpha^2}{2n^2}$ in terms of GM_EM_S and <r> for n large.

D. Use the observational fact that <r> = 1.5 × 10^{11}m to calculate the quantum number n. You will need $M_E = 6 \times 10^{24} kg$, $M_S = 2 \times 10^{30}$ kg, $G = 6.7 \times 10^{-11}$ Nm^2kg^{-2}, and $\hbar = 10^{-34}$Js.

E. The uncertainty in position is $\Delta r \equiv \sqrt{<r^2> - <r>^2}$. Calculate $\Delta r/<r>$ and express the result in terms of n.

2. A system of spin one particle is in a pure $S_z = +\hbar$ state, e.g. if a measurement of S_z is made, it will always yield the value $+\hbar$. Now if we make a measurement of S_x, what possible values will we get and what is the probability of each? Repeat the same calculation if we make a measurement of S_x^3.

[For a system of spin one particle, in a representation with S_z diagonal

$$S_z = \hbar\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix} \qquad S_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$].

3. A particle of mass m moves in a potential $V = \frac{1}{2}m\omega^2 x^2$. The normalized energy eigenfunctions are

$$u_n(x) = N_n e^{-\beta^2 x^2/2} H_n(\beta x) \quad \beta = \sqrt{\frac{m\omega}{\hbar}} \quad N_n = \sqrt{\frac{\beta}{\sqrt{\pi}2^n n!}}$$

The wave function at t = 0 is a displaced Gaussian:

$$\psi(x,0) = N_0 e^{-\beta^2 (x-b)^2/2}.$$

A. If the energy is measured, what is the probability of finding the result $\left(n+\frac{1}{2}\right)\hbar\omega$?

B. Compute <x> = $\int_{-\infty}^{\infty} dx \; \psi(x,t) x \; \psi(x,t)$. (Note: The final form of the answer should be very simple.)

$$[\int_{-\infty}^{\infty} d\xi \; e^{-\xi^2 + a\xi} \; H_n(\xi) = \sqrt{\pi} a^n e^{\frac{a^2}{4}}$$

$$\int_{-\infty}^{\infty} d\xi e^{-\xi^2} H_n(\xi) \xi H_m(\xi) d\xi = \sqrt{\pi} 2^{n-1} n! \; \delta_{m,n-1} + \sqrt{\pi} 2^n (n+1)! \delta_{m,n+1} \;]$$

4. Two identical spin $\frac{1}{2}$ particles of mass m are confined in a one dimensional infinite square potential well

$$V(x) = 0 \qquad -a < x < a$$
$$\quad\;\; = \infty \qquad \text{elsewhere}$$

Find the two lowest energy levels E_0, E_1 and their corresponding wave functions, state clearly the degrees of their degeneracy.

If the system is subject to a small perturbation

$$V' = c x_1 x_2$$

using a first order perturbation theory, determine the perturbed energies on these states.

$$\left[\int_{-a}^{a} \left(\cos\frac{\pi x}{2a} \sin\frac{\pi x}{a} \right) x dx = \frac{32a^2}{9\pi^2} \right]$$