QUANTUM MECHANICS

Do any 3 of the 4 problems.

1. Consider the infinite "delta comb" potential

$$V(x) = -g \sum_{n=1}^{\infty} \delta[x - (2n-1)a] - g \sum_{n=1}^{\infty} \delta[x + (2n-1)a] \quad ,$$

where the well strength g and the distance scale a are both > 0.

(a) Prove that this potential has at least one bound state, and derive the condition which determines its energy. (Hint: it may be helpful to employ the dimensionless parameter $\alpha = amg/\hbar^2$ at some point.)

(b) Show that it will have a second (nontrivial, or $E < 0$) bound state if ag is larger than some critical value, and find that critical value.

(c) If $g = \hbar^2/(2am)$, estimate the energies of all the bound states.

2. Consider a free particle of mass m constrained to move along the x axis, with the potential V equal to zero everywhere.

(a) Solve the time-independent Schrödinger equation for $\psi(x)$, subject to the periodic boundary condition $\psi(x) = \psi(x + L)$. Normalize ψ over the basic segment of length L.

(b) What are the energy eigenvalues? Qualitatively, compare the energy-level spectrum with that for a free particle not subject to periodic boundary conditions. Comment of the degeneracy (if present) of the energy states.

(c) Now suppose a weak (g is small) delta-function potential well $-g\delta(x)$ is placed at the origin. Treat this as a perturbation, and thus use perturbation theory to find the first-order effect on the lowest energy state. Find the corresponding "good" wavefunctions (i.e., those that would have allowed you to use nondegenerate perturbation theory), if applicable.

(d) Find the first-order effect on the next higher energy state. Find the "good" wavefunctions, if applicable. Are they what you expected? Why?

3. Consider a particle of mass m in the 1-D linear potential $V(x) = g|x|$.

(a) Sketch the ground state and the first excited state wavefunctions $\psi_0(x)$ and $\psi_1(x)$.

(b) Now solve the time-independent Schrödinger equation to show that the ground state and first excited state energies E_0 and E_1 are given by

$$E_0 \approx 0.8086 \left(\frac{g^2\hbar^2}{m}\right)^{1/3} ,$$

$$E_1 \approx 1.8558 \left(\frac{g^2\hbar^2}{m}\right)^{1/3} .$$

(The numerical factors out in front are very accurate, and you just need to get values close to these.) Hint: You might want to employ the variable $z = \alpha(gx - E)$, where α is a parameter you will need to determine; you may also employ the information on the following pages if needed.

(c) In a crude model. the "charmonium" atom consists of a charmed quark and antiquark, bound by the above 1-D potential. (This two-body problem reduces to an effective one-body problem for the reduced mass $m = m_c/2$, where m_c is the charmed quark mass.) Noting that the two lowest charmonium atom states have energies of 3.1 GeV and 3.7 GeV, respectively, determine the reduced mass of the charmed quark-antiquark pair mc^2 and the magnitude of g (in GeV/fm and N), which is the strong nuclear force between these particles. Is the magnitude of g (in N) reasonable, in light of other estimates of the strength of the strong force you've seen? Notes: Ignore relativistic effects in the binding, except to recognize that the energy of charmonium consists of the rest energy of the reduced-mass particle plus its kinetic and potential energy. Don't be bothered by the resulting charmed quark mass; it's a consequence of having a positive binding potential. The value of g is fine though. Finally, it is useful to remember that $\hbar c \approx 0.2\,\mathrm{GeV} \cdot \mathrm{fm}$.

4. Consider a simple 2-dimensional model of the hydrogen atom, where the electron moves in the x-y plane about the nucleus located at the origin, subject to the usual Coulomb potential $V(r) = -e^2/(4\pi\epsilon_0 r)$.

(a) What direction does the orbital angular momentum $\boldsymbol{L}$ point? Write down the operator for the orbital angular momentum of the electron.

(b) Write down the Hamiltonian H in the appropriate (i.e., cylindrical) coordinates r and ϕ, and then express it in terms of the angular momentum operator from above (along with some other terms of course).

(c) Does H commute with the angular momentum operator? What is the significance of this?

(d) Find the angular momentum eigenfunctions Φ and eigenvalues m_ℓ, such that $L_z\Phi = m_\ell\hbar\Phi$. Using this result, derive the equation which determines the radial part $R(r)$ of the wavefunction $\psi(r,\phi)$. This equation will then yield a (principal) quantum number n.

(e) Does the z component of spin angular momentum S_z commute with H and $\boldsymbol{L}$? What does this mean? What are the eigenvalues $m_s\hbar$ and eigenstates χ of S_z?

(f) Construct the total wavefunction for the system as $R(r)\Phi(\phi)\chi$, where the pieces have the associated quantum numbers n, m_ℓ, and m_s, repsectively. If you were to go through the details (do not do it), you would find that $n = 0, 1, 2, \ldots$ and $m_\ell = 0, \pm1, \pm2, \ldots, \pm n$. Suppose now the spin orbit interaction $H' = \alpha \boldsymbol{S}\cdot\boldsymbol{L}$ is included. Does $\boldsymbol{L}$ and S_z commute with the Hamiltonian now? What does this mean for the conservation of L_z and S_z? Find the perturbation to the state (n, m_ℓ, m_s). Is there a nonzero perturbation to the ground state $n = 0$? Contrast with the 3-D hydrogen atom.