1996

Quantum Mechanics

Do 3 of the 4 questions

1. Suppose a spherical delta function shell scattering potential of the form:

$$V(r) = \alpha\delta(r - a).$$

In the low energy regime $(ka << 1)$, and with α and a constants, calculate the scattering amplitude, the differential cross-section and the total cross section.
HINT: The wave function for $r > a$ may be written using the Rayleigh expansion as:

$$\Psi(r,\theta) = A\sum_{\ell=0}^{\infty}\left[i^{\ell}(2\ell+1)j_{\ell}(kr) + \sqrt{\frac{2\ell+1}{4\pi}}\,C_{\ell}h_{\ell}^{(1)}(kr)\right]P_{\ell}(\cos\theta),$$

where j_{ℓ} is the spherical Bessel function of order ℓ and the general solution contains the spherical Neumann function n_{ℓ} as well:

$$u(r) = A\, r j_{\ell}(kr) + B\, r n_{\ell}(kr)$$

$h_{\ell}^{(1)}$ is the spherical Hankel function

$$h_{\ell}^{(1)}(x) = j_{\ell}(x) + in_{\ell}(x),$$

P_{ℓ} is the usual Legendre polynomial, and C_{ℓ} is the partial wave amplitude.
The wave function for $r < a$ may be written

$$\Psi(r,\theta) = \sum b_{\ell}j_{\ell}(kr)P_{\ell}(\cos\theta),$$

where b_{ℓ} is a constant.

$$j_{\ell}(x) = (-x)^{\ell}\left(\frac{1}{x}\frac{d}{dx}\right)^{\ell}\frac{\sin x}{x}$$

$$n_{\ell}(x) = -(-x)^{\ell}\left(\frac{1}{x}\frac{d}{dx}\right)^{\ell}\frac{\cos x}{x}$$

$$P_o(x) = 1 \qquad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_1(x) = x \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

2. Write down the Hamiltonian for a neutral atom consisting of a heavy nucleus with charge Ze surrounded by Z electrons. Now simplify the expression by considering the atom to be Helium. Now ignore the Coulomb repulsive force between electrons and write down the simplest form of the Helium spatial wave function. What is the ground state spatial wave function for Helium? It is known experimentally that the ground state of Helium is a singlet. Why does this have to be the case? Excited states may be either singlet or triplet. What are the state variables for each?

3. Use the WKB approximation in the form

$$\int_{r_1}^{r_2} p(r)dr = \left(n - \frac{1}{2}\right)\pi\hbar$$

to estimate the bound state energies for Hydrogen. Under what condition do you recover the Bohr energy levels?

[You may arrive at an integral of the form:

$$\int_{r_1}^{r_2} \frac{\sqrt{-r^2 + Ar - B}}{r} dr,$$

where r_1 and r_2 are roots of the quadratic, so $-r^2 + Ar - B = (r - r_1)(r_2 - r)$.]

and $\int_{r_1}^{r_2} \frac{\sqrt{(r - r_1)(r_2 - r)}}{r} dr = \frac{\pi}{2}\left(\sqrt{r_2} - \sqrt{r_1}\right)^2$

4. Begin with the classical Lorentz force law for a charge particle moving with velocity v through an electromagnetic field and construct the Schrodinger equation for the particle in terms of the magnetic vector potential $\vec{A}$ and the scalar electric potential Φ. Find the expectation value of the velocity of the particle in terms of its momentum and the magnetic vector potential.

You may assume the "classical" Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\Phi$$

and for any observable $Q(x, p, t)$

$$\frac{d}{dt} < Q > = \frac{i}{\hbar} < \left[\hat{H}, \hat{Q}\right] > + < \frac{d\hat{Q}}{dt} >$$