QUANTUM MECHANICS

Answer any 3 of the 4 questions.

1. (a) Consider a particle of mass m in a one-dimensional delta-function potential $V = -\alpha\delta(x)$. Show from the Schrodinger equation that the wavefunction has a discontinuous gradient at $x = 0$, and further show from symmetry arguments that the left and right derivatives at $x = 0$ are

$$\psi'_{\pm} = \mp\frac{m\alpha}{\hbar^2}\psi(0).$$

Hence derive the form of the wavefunction and derive the value of the ground state energy.

(b) Now consider the use of a variational method to derive an approximation to the above solution. Start with a trial wavefunction of Gaussian form:

$$\psi(x) = ae^{-bx^2}$$

and minimize the expectation value of the total energy $T + V$ with respect to the parameter b. Compare your answer to the exact solution above.

2. The operator identifications $\hat{x} = x$, $\hat{p} = -i\hbar\,\partial/\partial x$ are only one of an entire set of possible representations. The fundamental requirement on any set is that they satisfy the commutation relation $[x, p] = i\hbar$.

(a) Verify that the coordinate representation above indeed does satisfy the commutation relation.

(b) If we instead identify $\hat{p} = p$ (a momentum representation), find the corresponding form of the operator $\hat{x}$.

(c) Consider the Hamiltonian for a one-dimensional simple harmonic oscillator, and show how appropriate identifications of the units for mass and time can reduce it to the form

$$H = \hat{p}^2 + \hat{x}^2$$

Write this Hamiltonian in both coordinate and momentum representations and comment on the symmetry between x and p that results. Hence show that the wavefunction solutions to the simple harmonic oscillator problem preserve their form under a Fourier transformation from coordinate to momentum space. Verify this explicitly for the ground state wavefunction.

3. (a) A spin $\frac{1}{2}$ particle is measured to have $S_x = \frac{1}{2}\hbar$ at time $t = 0$. It encounters a magnetic field $\mathbf{B} = B_0(\hat{y} + \hat{z})$ at time $t = t_0$. Derive the spin state for all times $t > t_0$.

(b) What is the probability of observing the spin in the $-\hat{x}$ direction at time $t > t_0$? What is the expectation value $\langle S_x^2 \rangle$?

The Pauli spin matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4. Consider two charged particles of equal mass m, and equal and opposite charges $\pm q$, confined to a one dimensional well $V = \frac{1}{2} m \omega_0^2 x^2$ and subject to an external electric field $E = E_0 \cos\omega t \, (\omega \neq \omega_0)$.

(a) Obtain an expression for the Hamiltonian of the system in center of mass coordinates and use it to derive the equations of motion for the expectation values of the positions of the two particles in the center-of-mass frame.

(b) Obtain the expectation value of the relative distance between the two particles, assuming that they are localized at $x = \pm x_0$, with the same momentum expectation value, at $t = 0$.

You may assume the result:

$$\frac{d}{dt}\langle A \rangle = \frac{i}{\hbar}\langle [H, A] \rangle + \langle \partial A / \partial t \rangle$$