Department of Physics

University of Alabama in Huntsville
Comprehensive Examination 2010

Quantum Mechanics

Name: ____________________________________

Answer three (3) questions - Circle and clearly indicate final answers

1. **Bag model for nucleons.** A nucleon is made of three almost massless quarks, but the rest mass of a nucleon is about 938 MeV. The quarks are confined to the interior of nucleons because it takes work to disturb the vacuum and create a region of space in which they can be present. The work that must be done in order to make a space in which quarks can live is parametrized by a constant known as the "bag constant" B, which has units of energy per unit volume. That is, the work needed to make a "quark bag" or nucleon of volume V is BV .

   A simple model of the nucleon treats it as a spherical bag of radius R, with a rest energy (i.e., rest mass) that includes only two terms: (i) the work done to create the bag, and (ii) the zero point kinetic energy of the massless quarks confined within the bag. A massless particle has an energy $E = pc$, and from the uncertainty principle we can set $p = \hbar/R$. (Note: this is the usual non-relativistic result; however, allowing the quarks to be relativistic yields essentially the same answer, so we will just go ahead and use the non-relativistic expression.) So, the rest energy of the nucleon E_{tot} can be written as the sum of the zero point kinetic energy and the bag energy, or

$$E_{\text{tot}} = 3\frac{\hbar c}{R} + B\frac{4}{3}\pi R^3 \quad .$$

   In this model, the radius of the nucleon R_0 is the value of R that minimizes E_{tot}.

   (a) Derive an expression for R_0 and the corresponding value of E_{tot}. Check your work by verifying units. (Note: $\hbar c = 197$ MeV fm. You will waste a massive amount of time by converting everything to SI units! Use this value of $\hbar c$ and work with MeVs and fermis.)

   (b) With $E_{\text{tot}} = 938$ MeV, find the value of B in units of MeV/fm^3, and then atmospheres. This is the pressure that the quarks must fight against to open up the bag and escape the nucleon. Also find the value of R_0.

   (c) Suppose you wanted to begin to investigate this model taking into account the quark mass. For a quark of mass m, find the quantum mechanical (i.e., nonrelativistic) ground state energy E_1 in an infinite spherical well (the bag) of radius R. (Note: Using this expression to refine your approach above would prove to be unfruitful. You need the relativistically-correct expression. However, since this is an exam on quantum mechanics, just find the non-relativistic expression.)

2. **Polarizability of a particle on a ring.** (Most of the credit is for parts (a) and (b); do (c) and (d) later if you have time.) Consider a particle of mass m constrained to move in the $x - y$ plane on a circular ring of radius a, centered on the origin.

   (a) Recalling that the Hamiltonian can be written as $H^0 = L_z^2/2ma^2$, calculate the eigenvalues E^0 and (properly normalized) eigenfunctions ψ^0 of H^0. Comment on degeneracy.

   (b) Now assume that the particle has a charge q and that it is placed in a uniform electric field ε in the x direction. This introduces a perturbation $V = -q\varepsilon a \cos\varphi$. Find the energies E^1 of the new levels to first order in the electric field. A useful integral is

$$\int_0^{2\pi} e^{-i2m\varphi} \cos\varphi \, d\varphi = 0 \quad ,$$

   for any integer m.

   (c) Find the perturbed ground state wavefunction to first order in the electric field. A useful integral is

$$\int_0^{2\pi} e^{-im\varphi} \cos\varphi \, d\varphi = 0 \quad ,$$

   for integer $m = \pm 2, \pm 3, \pm 4, \ldots$ but equal to π for $m = \pm 1$.

   (d) Use the new wavefunction from part (c) to find the induced electric dipole moment $\langle \psi | qx | \psi \rangle$. Determine the proportionality constant between the dipole moment and the applied electric field. (This proportionality constant is the polarizability of the system.)

3. **S-wave scattering from a delta function potential.** Consider S-wave ($\ell = 0$) scattering from the potential

$$V(r) = \lambda \frac{\hbar^2}{2mR} \delta(r - R),$$

where λ is a positive constant. To find the phase shift δ_0, we have to solve

$$\frac{d^2u}{dr^2} + k^2 u = \frac{\lambda}{R} \delta(r - R) u,$$

with $u = 0$ at $r = 0$ and $u = B \sin(kr + \delta_0)$ for $r > R$.

(a) What is u for $r < R$?

(b) Apply the appropriate conditions on u and du/dr at $r = R$ to find an equation that determines δ_0.

(c) Find the scattering length a, defined by $\lim_{k\to 0} \delta_0 = -ka$.

(d) On physical grounds, what behavior would you expect to see in the S-wave cross section σ_0? At what values of k?

4. **Delta function potential.** Suppose a delta-function well $-g\delta(x-a)$ is placed a distance a to the right of an infinite potential barrier at $x < 0$.

   (a) Find the transcendental equation that determines the bound state energy(ies) for a particle of mass m confined to the x axis with this potential. Express your result in terms of the dimensionless variables $y = 2a\kappa$ and $c = \hbar^2/2mag$.

   (b) Will there always be a bound state? If no, what is the condition for one to exist?