Quantum Mechanics

1.

a. Derive the formula relating probability density $\rho(\vec{r},t)$ and probability current $\vec{j}(\vec{r},t)$,

$$\frac{\partial}{\partial t}\rho + \nabla \bullet \vec{j} = 0 \tag{1}$$

from the Schrödinger equation.

b. If the potential is complex, what is the new form of Eq. (1)?

c. What is $\vec{j}$ for a hydrogen atom in state (n, l, m)?

2.

a. Write down the spin wave functions for two s = 1/2 particles with $H_0 = \frac{J}{\hbar^2}\vec{S}_1 \bullet \vec{S}_2$. What is the energy level of each state?

b. Obtain the wave functions and energy levels for

$$H = \frac{J'}{\hbar^2} S_{1x}S_{2x} + \frac{J}{\hbar^2}\left(S_{1y}S_{2y} + S_{1z}S_{2z}\right)$$

when $J' \neq J$ but $\Delta J = J' - J << J$, so that this is a perturbation of the situation in (a). Note: $S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. Show that the probability for a 1s electron to be found inside the nucleus is approximately 1.5 x 10^{-4} Z, where the wave function has the form $\psi_{100}(r,\theta,\varphi) = Ae^{-Zr/a_0}$. Z is the atomic number and $a_0 = 5.29x10^{-11}$ m is the Bohr radius; assume a nuclear radius of 4 x 10^{15}m.

$$\text{Hint: } \int r^n e^{-\alpha r} dr = (-1)^n \frac{d^n}{d\alpha^n}\int e^{-\alpha r} dr$$

4. For times t < 0, a two-level system has the Hamiltonian $H_0 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$ and is in the state with energy +E. At t = 0, the Hamiltonian is suddenly changed to $H = \begin{pmatrix} E & E \\ E & -E \end{pmatrix}$ (note that this change is too great to be considered a perturbation).

   a. What are the possible results of an energy measurement at t = 0? What is the probability of each possible result? What is the expectation value of the energy at t = 0?

   b. Given $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, what is <A> at $t = \dfrac{\pi\hbar}{2\sqrt{2}E}$?