QUANTUM MECHANICS

Do any 3 of the 4 problems.

1. Consider an infinite, one-dimensional square well extending from $x = 0$ to $x = 2w$. Inside the well, the potential is zero for $0 < x < w$ but V_0 for $w \le x < 2w$ (i.e., there is a potential "shelf" inside the well).

   (a) Derive the condition that yields the bound state energies for $E < V_0$. Express this condition in terms of $\theta = w\kappa = w(\sqrt{2m(V_0 - E)}/\hbar)$ and $\phi = wk = w(\sqrt{2mE}/\hbar)$.

   (b) There is a critical value of V_0 such that, below this value, there is no bound state with $E < V_0$. Find an approximate value for this critical potential shelf height. (Hint: this is easier if you normalize energies to the natural energy scale of the problem.)

Possibly (depending upon method) Useful Information for Problem 1

$$\frac{2A}{\sqrt{k(x)}}\cos\left(\int_x^a k(x)\,dx - \frac{\pi}{4}\right) - \frac{B}{\sqrt{k(x)}}\sin\left(\int_x^a k(x)\,dx - \frac{\pi}{4}\right) \longleftrightarrow$$
$$\frac{A}{\sqrt{\kappa(x)}}\exp\left(-\int_a^x \kappa(x)\,dx\right) + \frac{B}{\sqrt{\kappa(x)}}\exp\left(\int_a^x \kappa(x)\,dx\right)$$

2. In this problem, you'll calculate the first-order correction to the ground state energy of a H atom due to the finite size of the nucleus. Assume that a proton is a uniformly-charged sphere of radius R centered on the origin.

   (a) Find the potential energy of an electron both inside and outside the uniformly-charged proton sphere.

   (b) Derive an approximate expression for the first-order correction to the ground-state energy, taking $R = 1\,\text{fm}$. The unperturbed ground-state wavefunction is $\psi_1^0 = (\pi a^3)^{-1/2}\exp(-r/a)$, where a is the Bohr radius. What is the size of this correction in eV?

3. Consider a one-dimensional, infinite square well extending from $x = -a$ to $x = b$. Now suppose a delta-function potential $\alpha\delta(x)$ is added at the origin, where $\alpha > 0$.

(a) Derive the equation the gives the bound state energies.

(b) If a and b are both small, show that only one bound state exists and calculate its energy.

4. Consider a particle of mass m in a one-dimensional, simple harmonic oscillator potential $V(x) = m\omega^2 x^2/2$.

(a) Find the expectation values of kinetic and potential energy in stationary state n.

(b) Derive a differential equation for the time evolution of the expection value $\langle x \rangle$. Now solve this equation, assuming that the initial expectation value of momentum is $\langle p_x \rangle_0$ and that the initial expectation value of position is $\langle x \rangle_0$. Is this behavior to be expected?

(c) Now suppose that $V(x) \to +\infty$ for $x < 0$, but retains its SHO shape for $x > 0$ (corresponding to, say, a spring that can be stetched but not compressed). Find the allowed energy levels.