QUANTUM MECHANICS

Do any 3 of the 4 problems. Circle and clearly indicate final answers.

1. The dynamics of a two-state system (such as the ammonia molecule, or the double harmonic oscillator) are well known. Here is a more interesting two-state system, in which the states may also decay (M. Gell-Mann and A. Pais, Phys. Rev., 97, 1387 (1955)).

The neutral kaon is a system with two basis states, the kaon $|K^0\rangle$ and its antiparticle $|\bar{K}^0\rangle$. The combined charge conjugation and parity operation CP takes the state $|K^0\rangle$ into the state $|\bar{K}^0\rangle$ and vice versa, and is therefore the particle-antiparticle transformation with a matrix representation of σ_x. The dynamics of this system is governed by the Hamiltonian

$$H = M - i\frac{\Gamma}{2},$$

where M and Γ are Hermitian 2×2 matrices, representing the mass energy and decay properties of the system, respectively. The matrix M is positive definite, and a fundamental symmetry (combined CP and time-reversal) requires that $\sigma_x M^* = M\sigma_x$ and $\sigma_x \Gamma^* = \Gamma\sigma_x$. Assuming (as is the case to a good approximation) that H also satisfies CP invariance, we have additionally that $\sigma_x M = M\sigma_x$ and $\sigma_x\Gamma = \Gamma\sigma_x$. Hence, $M_{11} = M_{22} = a$, $M_{12} = M_{21} = b$, $\Gamma_{11} = \Gamma_{22} = c$, and $\Gamma_{12} = \Gamma_{21} = d$, where a, b, c, and d are real.

(a) Show that H is a normal operator, and find its eigenvalues and (properly normalized) eigenstates $|K_1^0\rangle$ and $|K_2^0\rangle$. (The second eigenstate corresponds to the eigenvalue with the larger real part.) (Do not leave the eigenstates in matrix form; express them in terms of the basis states.)

(b) The lifetimes τ_0^1 and τ_0^2 of the two eigenstates, as well as their mass difference $(m_2 - m_1)c^2 > 0$ are well-measured quantities. If the lifetime of a state is defined by the time dependence $\exp(-t/2\tau)$, express as many matrix elements as possible in terms of these measured quantities.

(c) Suppose that a K^0 is produced in a high-energy collision at time $t = 0$. What is the probability that it will be found in the state $\bar{K}^0$ at time t? (Note the particle-antiparticle oscillations, superposed upon an overall decay.)

2. Consider a particle of mass m and an infinitely-long delta-function potential cylinder: that is, in cylindrical coordinates (r, ϕ, z), the potential energy $V = -aV_0\delta(r - a)$, where $V_0 > 0$.

(a) Derive the condition that yields the bound-state energy(ies) E for arbitrary angular momentum quantum number m. Express this condition in terms of the dimensionless characteristic potential energy $\alpha = 2ma^2V_0/\hbar^2$.

(b) Now suppose that $V \to +\infty$ for $r \geq a$, while V remains zero inside the cylinder ($r < a$). Find the bound state energy(ies).

3. This is an alternative and simpler derivation of the (Landau) energy levels of a particle in a magnetic field. Consider an electron of mass m and charge $-e < 0$ in a uniform magnetic field $\boldsymbol{B} = B\widehat{\mathbf{z}}$.

(a) Find the vector potential $\boldsymbol{A}$ in the symmetric gauge, where $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{r}/2$.

(b) Express the Hamiltonian H as $H_z + H_{xy}$, which depend only upon z and $x - y$ motion, respectively.

(c) Further express H_{xy} as $H_1 + H_2$, where H_2 is proportional to L_z. Write H_1 and H_2 in terms of the Larmor frequency $\omega = eB/2mc$, which is one half of the electron cyclotron frequency ω_c.

(d) It can be shown that $[H_{xy}, H_z] = 0$. Argue why H_1 commutes with H_2. Hence, there exist simultaneous eigenstates of H_z, H_1, and L_z. (It would be tempting to thus identify 3 components of the total energy, but this is not correct, since L_z depends on the gauge.) We need another way of finding the eigenenergies of H_{xy}...

(e) Define $a_\pm^\dagger = (a_x^\dagger \pm i a_y^\dagger)/\sqrt{2}$, where the $a_j^\dagger$ and a_j are the usual raising and lowering operators for component $j = x, y$. Show that H_1 can be written as $\hbar\omega(a_+^\dagger a_+ + a_-^\dagger a_- + 1)$.

(f) L_z can be written as $\hbar(a_+^\dagger a_+ - a_-^\dagger a_-)$. Find the eigenenergies of H_{xy} (the energy associated with motion in the $x - y$ plane) and then of the whole Hamiltonian.

4. Suppose a charge q attached to a spring and constrained to oscillate along the x axis starts out in the state $|n\rangle$.

(a) Write down the wavefunction and energy of this state.

(b) Now suppose that the state decays via spontaneous emission to a state $|n'\rangle$. What is the general condition, qualitatively, that determines when spontaneous emission dominates stimulated emission in a system at thermal equilibrium with its surroundings?

(c) Calculate the dipole moment for the transition. What transitions are allowed (in terms of n' and n)?

(d) What is the rate at which energy is radiated during the transition? Compare with the classical rate $P = q^2\omega^2 E_n/6\pi\epsilon_0 mc^3$ (obtained from the Larmor formula), where E_n is the energy of the initial state. Comment. (The Einstein A coefficient is given by $\omega^3|\boldsymbol{p}|^2/3\pi\epsilon_0\hbar c^3$ (SI units).)