QUANTUM MECHANICS

(Work 3 of the 4 problems)

1. The time–independent Schrodinger equation for the harmonic oscillator is

$$H|n\rangle = E_n|n\rangle = (n + \tfrac{1}{2})\hbar\omega|n\rangle,$$

where $|n\rangle$ is the state of energy E_n and the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 .$$

In H, p and x are the position and momentum operators, which may be used to define the creation and annihilation operators,

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left[x - i\frac{p}{m\omega}\right],$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left[x + i\frac{p}{m\omega}\right],$$

which satisfy the commutator relation $[a, a^+] = 1$, and whose action is given by $a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a|n\rangle = \sqrt{n}\,|n-1\rangle$.

(a) Use the creation and annihilation operators to show that $\Delta x\Delta p = \left[n + \frac{1}{2}\right]\hbar$ in the state $|n\rangle$. Recall that $\Delta Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2}$ for an operator Q, and that $\langle Q\rangle = \langle n|Q|n\rangle$ in the state $|n\rangle$.

(b) Use the fact that the annihilation operator destroys the ground state, i.e., $a|0\rangle = 0$, to show that the ground state wave function is of the form

$$\phi_0(x) = A\, e^{-\frac{m\omega x^2}{2\hbar}} ,$$

where A is a constant.

2. An electron (spin $\frac{1}{2}$) is in a uniform magnetic field $\vec{B} = B_0\, \hat{e}_z$. At time $t = 0$ the spin is pointing in the x–direction, i.e., $<S_x(t = 0)> = \frac{\hbar}{2}$. Find the expectation value $<\vec{S}(t)>$ for arbitrary t.

3. The quantum system consisting of neutral kaons and their antiparticles can be described by a two–dimensional state space spanned by the orthonormal basis of strangeness eigenvectors $\{|K^\circ>, |\bar{K}^\circ>\}$.

(a) The particles K° and $\bar{K}^\circ$ have strangeness +1 and −1, respectively; if S is the strangeness operator, $S|K^\circ> = |K^\circ>$, $S|\bar{K}^\circ> = -|\bar{K}^\circ>$. What is the matrix form of S in this basis, where $|K^\circ> = \begin{bmatrix}1\\0\end{bmatrix}$ and $|\bar{K}^\circ> = \begin{bmatrix}0\\1\end{bmatrix}$?

(b) In this same basis, the charge conjugation operator is

$$C = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} .$$

Where are $C|K^\circ>$ and $C|\bar{K}^\circ>$?

(c) Find the eigenvalues and eigenvectors ($|K_L>$ and $|K_S>$) of C.

(d) Find the transformation operator U (matrix form) that transforms from the $\{|K^\circ>, |\bar{K}^\circ>\}$ basis to the $\{|K_L>, |K_S>\}$ basis.

(e) Use U to find the matrix form of S in the $\{|K_L\rangle, |K_S\rangle\}$ basis.

4. Two spin $-\frac{1}{2}$ particles, each of mass m, are confined within a one–dimensional box (infinite square well) of width a. The Hamiltonian is:

$$H_o \begin{cases} \frac{P_1{}^2}{2m} + \frac{P_2{}^2}{2m}, & \text{inside well} \\ \infty & \text{, outside} \end{cases}$$

(a) Assume that the particles are <u>distinguishable</u> and do not interact. Give the energies and degeneracies of the lowest three energy levels of the two–particle system.

(b) Now assume that the particles are <u>indistinguishable</u> (identical) but still do not interact. Given the energies and degeneracies of the lowest three energy levels of the two–particle system.

(c) Finally, assume that the particles are still indistinguishable, but now interact via a repulsive delta–function potential:

$$\begin{cases} \frac{P_1{}^2}{2m} + \frac{P_2{}^2}{2m} + aV_o\delta(x_1 - x_2), & \text{inside well} \\ \infty & \text{, outside} \end{cases}.$$

Treat the repulsive ($aV_o > 0$) interaction as a perturbation, and apply it to the states comprising the most degenerate level of part (b). Note that the perturbation is diagonal in the basis made up of these states, so ordinary <u>non–degenerate perturbation theory applies</u>. Show <u>qualitatively</u> that the first–order corrections to the energies of these states agree with Hund's rule that says that the electrons in a partially filled atomic subshell tend to align their spins (i.e., the state with the greatest total spin will have the lowest energy).