The Gaussian Correlation Conjecture

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The GCC

Conjecture

1. *For two symmetric, convex sets in* $\mathbb{R}^n$, K *and* L *and* μ *a mean zero Gaussian measure on* $\mathbb{R}^n$

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

2. *For any* $s, t \geq 0$*, norms* $|||\cdot|||, \|\cdot\|$ *and* G *standard Gaussian on* $\mathbb{R}^n$*, we have*

$$\Pr(|||G||| \leq s, \|G\| \leq t) \geq \Pr(|||G||| \leq s) \Pr(\|G\| \leq t)$$

3. *By complementation:*

$$\Pr(|||G||| > s, \|G\| \leq t) \leq \Pr(|||G||| > s) \Pr(\|G\| \leq t)$$

Known Results

An old result of Sidak is

Theorem

If K is a symmetric convex set in $\mathbb{R}^n$ and $b \in \mathbb{R}^n$, then

$$\mu(K \cap \{|\langle X, b\rangle| \leq t\}) \geq \mu(K)\mu(|\langle X, b\rangle| \leq t)$$

More generally, we have a result of Hargé .

Theorem

If K is a symmetric convex set in $\mathbb{R}^n$ and $\mathcal{E}$ is a symmetric ellipsoid in $\mathbb{R}^n$, then

$$\mu(K \cap \mathcal{E}) \geq \mu(K)\mu(\mathcal{E})$$

Theorem

(Corollary of a result of Yaozhong Hu) For any two norms,

$$\mathbb{E}\|G\| \cdot |||G||| \geq \mathbb{E}\|G\| \; \mathbb{E}|||G|||.$$

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- Let $\|\cdot\|_1$ denote the ℓ_1^n norm on $\mathbb{R}^n$.
- Let $G \sim N(0, I_n)$
- $V : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for some appropriately chosen orthogonal transformation such that

$$\Pr(\|G\|_1 \leq \mathbb{E}\|G\|_1, \|VG\|_1 \leq \mathbb{E}\|G\|_1)$$

$$<< \Pr^2(\|G\|_1 \leq \mathbb{E}\|G\|_1) \approx (\frac{1}{2})^2$$

Note that

$$\{\|G\|_1 \le \mathbb{E}\|G\|_1\} = \{\sum_{j=1}^{n}(|g_j| - \mathbb{E}|g|) \le 0\}$$
$$= \{\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(|g_j| - \mathbb{E}|g|) \le 0\}$$

And similarly,

$$\{\|VG\|_1 \le \mathbb{E}\|G\|_1\} = \{\sum_{j=1}^{n}(|h_j| - \mathbb{E}|h|) \le 0\}$$
$$= \{\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(|h_j| - \mathbb{E}|h|) \le 0\}$$

The CLT applies in each case and these marginal probabilities converges to

$$\Pr(N(0, 1 - \frac{2}{\pi}) \leq 0)$$

In particular, the pair $(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} (|g_j| - \mathbb{E}|g|), \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (|h_j| - \mathbb{E}|g|))$ is tight. Suppose we take a subsequence which converges in distribution.

- Then, each limiting coordinate is the above normal. Let call such a limiting pair (ξ, η), then we can write

$$\eta = \sigma\xi + \xi^{\perp},$$

where $\sigma = \dfrac{\mathbb{E}\xi\eta}{\mathbb{E}(\xi)^2}$ and $\xi^{\perp}$ is orthogonal to and, hence, in the jointly normal case, independent of ξ.

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- Therefore, in the jointly normal case we have

$$\begin{aligned}\Pr(\xi \le 0, \sigma\xi + \xi^{\perp} \le 0) &= \mathbb{E}\Pr(\xi \le 0, \xi^{\perp} \le -\sigma\xi) \\ &= \mathbb{E} Pr(\xi^{\perp} \le -\sigma\xi \big| \xi) I_{\xi \le 0} \ge \mathbb{E}\Pr(\xi^{\perp} \le 0) I_{\xi \le 0} = \frac{1}{4}.\end{aligned}$$

What have we learned?

- While we don't know if we can assume joint normality, it was key to being able to use the fact that ξ and η were positively correlated, which allowed us to use the fact that $-\sigma\xi \geq 0$ on the set that $\xi \leq 0$.

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- Tracing this back to the step prior to taking a limit leads us to the question: Is

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- But,

$$\mathbb{E}\|VG\|_1 I_{\|G\| \leq \mathbb{E}\|G\|_1} = \sum_{j=1}^{n} \mathbb{E}|h_j| I_{\|G\|_1 \leq \mathbb{E}\|G\|_1}.$$

Some Success

Now, using Sidak's result plus complementation, we have for any $b > 0$ (think $b = \mathbb{E}\|G\|_1$)

$$\sum_{j=1}^{n} \mathbb{E}|h_j| I_{\|G\|_1 \le b} = \sum_{j=1}^{n} \int_0^\infty \Pr(|h_j| I_{\|G\|_1 \le b} > t)\, dt$$

$$= \sum_{j=1}^{n} \int_0^\infty \Pr(|h_j| > t, \|G\|_1 \le b)\, dt$$

$$\le (\text{Sidak}) \ \sum_{j=1}^{n} \int_0^\infty \Pr(|h_j| > t) \Pr(\|G\|_1 \le b)\, dt$$

$$= \sum_{j=1}^{n} \mathbb{E}|h_j| \Pr(\|G\|_1 \le b) = \mathbb{E}\|VG\|_1 \ \Pr(\|G\|_1 \le b)$$

$$= \mathbb{E}\|G\|_1 \ \Pr(\|G\|_1 \le b).$$

A New Question

This also raises the question: Is such an inequality always true? Namely, for two norms, $\|\cdot\|$ and $|||\cdot|||$ do we have

$$\mathbb{E}|||G||| I_{\|G\|\le b} \le \mathbb{E}|||G||| \Pr(\|G\| \le b)?$$

Of course, we might wonder if this is too much to hope for.

A Necessary Condition

But, if we assume that the GCC is true, then similar to the above calculation

$$\begin{aligned}
\mathbb{E}|||G|||I_{\|G\|\le t} &= \int_0^\infty \Pr(|||G|||I_{\|G\|\le t} > s)\,ds \\
&\ge \int_0^\infty \Pr(|||G||| > s)\Pr(\|G\| \le t)\,dt \\
&= \mathbb{E}|||G|||\Pr(\|G\| \le t).
\end{aligned}$$

So, the above condition is a necessary condition.

New General Questions

We now have more questions than we started with, namely,

Questions

1. *Is the pair* (ξ, η) *necessarily jointly normal?*
2. *Do we always have the* **truncated mean inequality**

$$\mathbb{E}\|G\| I_{\|G\| \le t} \le \mathbb{E}\|G\| \Pr(\|G\| \le t)?$$

What if we do have the truncated mean inequality holding?

Kwapien proved that for a Gaussian vector, G, in a topological vector space, F, and a continuous, convex function, f on F, one has

$$\text{med}(f(G)) \leq \mathbb{E}f(G).$$

He uses Ehrhard's inequality, which was mentioned by Wenbo. It may be possible to show something similar for a conditioned Gaussian in order to prove

$$\text{med}(\|G\| I_{\|G\| \leq b}) \leq \mathbb{E}\|G\| I_{\|G\| \leq b}.$$

Let's assume that both the truncated mean inequality holds and a conditioned Kwapien result holds . Then we have for $F = \{\|G\| \leq \mathbb{E}\|G\|\}$

$$
\begin{aligned}
\text{(1)} \qquad & \Pr(|||G||| \leq \mathbb{E}|||G|||, \|G\| \leq \mathbb{E}\|G\|) \\
& = \Pr\left(|||G||| \leq \mathbb{E}|||G||| \,\middle|\, F\right) \Pr(F) \\
\text{(2)} \qquad & \geq \Pr\left(|||G||| \leq \mathbb{E}\left(|||G||| \,\middle|\, F\right) \,\middle|\, F\right) \Pr(F)
\end{aligned}
$$

By Kwapien's theorem $\Pr(F) \geq \Pr(\|G\| \geq \text{med}(\|G\|)) \geq 1/2$. With our assumption that conditioned Kwapien holds, the other probability in (2) is also at least $1/2$. Hence, the probability of the intersection in (1) has asymptotically the correct value for the GCC.

What's up Doc?

- If we are trying to find a counterexample to the GCC, we might first try to find an example for which the truncated mean inequality fails (it holds for the ℓ_1^n-ball case).

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- If we are trying to find a counterexample to the GCC, we might first try to find an example for which the truncated mean inequality fails (it holds for the ℓ_1^n-ball case).
- Or, we can try to find an example for which the conditional Kwapien inequality fails.