Central limit theorem for an additive functional of the fractional Brownian motion

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Joint work with Yaozhong Hu and David Nualart

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Fractional Brownian Motion

d-dimensional fractional Brownian motion

$B_t = \left(B_t^1, \cdots, B_t^d\right)$ is a centered Gaussian process with stationary increments and covariance function

$$\mathbb{E}\,(B_t^i B_s^j) = \frac{\delta_{ij}}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

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- fBm has self-similar property: for any $a > 0$, $a^{-H} B_{at} \stackrel{d}{=} B_t$.
- If $H = \frac{1}{2}$, B_t is the Brownian motion.
- If $H \neq \frac{1}{2}$, B_t is neither a Markov process nor a semimartingale.

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- If $Hd < 1$, then the local time of B exists and can be defined as

$$L_t(x) = \int_0^t \delta(B_s - x) ds, \quad t \geq 0,\ x \in R^d\,,$$

where δ is the Dirac delta function. Moreover, $L_t(x)$ is jointly continuous with respect to t and x.

Central Limit Theorem

The scaling property of the fBm and the continuity of the local time immediately imply the following result.

Theorem

Suppose $Hd < 1$ and $f \in L^1(\mathbb{R}^d)$. Then

$$\left(n^{Hd-1}\int_0^{nt} f(B_s)\,ds\,,\,t\geq 0\right)\ \overset{\mathcal{L}}{\to}\ \left(L_t(0)\int_{\mathbb{R}^d} f(x)\,dx\,,\,t\geq 0\right).$$

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Remark

If $\int_{\mathbb{R}^d} f(x)dx = 0$, then $n^{Hd-1}\int_0^{nt} f(B_s)\,ds$ converges to 0.

Question

If $\int_{\mathbb{R}^d} f(x)dx = 0$, is there a $\beta > Hd-1$ such that $n^{\beta}\int_0^{nt} f(B_s)\,ds$ converges to a nonzero process?

Previous Work

Kallianpur and Robbins (1953)

Suppose $d = 2$ and $H = \frac{1}{2}$. For any bounded and integrable function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{1}{\log n} \int_0^n f(B_s) ds \xrightarrow{\mathcal{L}} \frac{Z}{2\pi} \int_{\mathbb{R}^2} f(x) dx,$$

as n tends to infinity, where Z is a random variable with exponential distribution of parameter 1.

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- If $\int_{\mathbb{R}^2} f(x)dx \neq 0$, then $\frac{1}{\log n} \int_0^{n^t} f(B_s)ds \xrightarrow{M_1} \frac{Z_t}{2\pi} \int_{\mathbb{R}^2} f(x)dx$.

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- If $\int_{\mathbb{R}^2} f(x)dx = 0$, then $\frac{1}{\sqrt{\log n}} \int_0^{n^t} f(B_s)ds \xrightarrow{M_1} C_f W(Z_t)$ where $C_f = (-\frac{2}{\pi} \int \int f(x)f(y) \log |x - y| dxdy)^{1/2}$.

Previous Work

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The Kallianpur-Robbins law was extended to the fBm by Kôno in 1995, and the corresponding functional version was obtained by Kasahara and Kosugi in 1997.

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Papanicolaou, Stroock and Varadhan (1977)

Suppose $d = 1$, $H = \frac{1}{2}$ and $\int_{\mathbb{R}} f(x)dx = 0$. Then

$$\Big(n^{-\frac{1}{4}} \int_0^{nt} f(B_s)\, ds\,,\ t \geq 0\Big) \xrightarrow{\mathcal{L}} \Big(C_f\, W(L_t(0))\,, t \geq 0\Big)$$

where $C_f = \big(-2\int_{\mathbb{R}^{2d}} f(x)f(y)|x-y|\,dx\,dy\big)^{1/2}$ and W is a real-valued standard Brownian motion independent of B.

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Remark

Papanicolaou, Stroock and Varadhan's result was generalized to stable processes by Rosen in 1991.

Notation

Fix a number $\beta > 0$ and denote

$$H_0^\beta = \left\{ f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} |f(x)||x|^\beta dx < \infty \ \text{ and } \int_{\mathbb{R}^d} f(x)\, dx = 0 \right\}.$$

For any $f \in H_0^\beta$,

$$\|f\|_\beta^2 := -\int_{\mathbb{R}^{2d}} f(x) f(y) |x-y|^\beta \, dx \, dy$$

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Local Nondeterminism Property, Berman (1973)

For any $0 = s_0 < s_1 \leq \cdots \leq s_n < \infty$ and $u_1, \ldots, u_n \in \mathbb{R}^d$, there exists a positive constant k_H such that

$$\mathrm{Var}\Big(\sum_{i=1}^n u_i \cdot (B_{s_i} - B_{s_{i-1}}) \Big) \geq k_H \sum_{i=1}^n |u_i|^2 (s_i - s_{i-1})^{2H}.$$

Central Limit Theorem

Hu, Nualart and Xu (submitted)

Suppose $\frac{1}{d+1} < H < \frac{1}{d}$ and $f \in H_0^{\frac{1}{H}-d}$. Then

$$\left(n^{\frac{Hd-1}{2}} \int_0^{nt} f(B_s)\, ds\,,\;\; t \ge 0\right) \quad \xrightarrow{\mathcal{L}} \quad \left(\sqrt{C_{H,d}}\, \|f\|_{\frac{1}{H}-d}\, W(L_t(0))\,, t \ge 0\right)$$

in the space $C([0,\infty))$, as n tends to infinity, where W is a real-valued standard Brownian motion independent of B and

$$C_{H,d} = \frac{2^{1-\frac{1}{2H}}}{1-Hd}\pi^{-\frac{d}{2}}\Gamma\left(\frac{Hd+2H-1}{2H}\right).$$

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Remark

- $C_{H,d}$ is finite for any $\frac{1}{d+2} < H < \frac{1}{d}$.
- We conjecture that our result also holds for $\frac{1}{d+2} < H < \frac{1}{d}$.

Thank You !