Quenched Asymptotic for Ornstein-Uhlenbeck process of Poisson Potential

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Random Motion in Random Media

- random motion: $\mathbb{R}^d$ valued Markovian process $X(t)$
  – P_x, $\mathbb{E}_x$
- random media: stationary random potential $V(\cdot)$
  –P, $\mathbb{E}$
- $X(t)$ and $V(\cdot)$ are independent.
- Consider

$$u_{\pm}(t,x) \stackrel{def}{=} \mathbb{E}_x \exp\left\{\pm \int_0^t V(X(s))\,\mathrm{d}s\right\}$$

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Why exponential moment?

- $u(t,x)$ is the normalizing constant of the random Gibbs measure $\mu_{t,\omega}$

$$\frac{\mathrm{d}\mu_{t,\omega}}{\mathrm{dP}_x} = \frac{1}{u(t,x)} \exp\left\{ \pm \int_0^t V(X_s)\,\mathrm{d}s \right\}$$

- (Feynman-Kac) $u(t,x)$ solves

$$\begin{aligned} \partial_t u(t,x) &= \mathcal{L}u(t,x) \pm V(x)u(t,x), \qquad && (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= 1, && x \in \mathbb{R}^d, \end{aligned}$$

$\mathcal{L}-$ infinitesimal generator of the Markov operator T_t
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Literature

Brownian motion under stationary potentials have been well studied:

- Sznitman ’93
- Carmona, Molchanov ’95
- Gartner, Konig, Molchanov ’00
- Gartner, Konig ’05
- Chen ’11

Our Focus

$X(t)$ be Ornstein-Uhlenbeck (O-U) process.

Why O-U?

- Extensive applications for stationary dynamics in other fields:
  - physical science: noisy relaxation process
  - finance: interest rate derivatives
  - biochemistry: model peptide bond angle of water molecules
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O-U process

Let $X(t)$ be a d-dim O-U process $X(t)$ starting at $x = (x_1, \ldots, x_d)$:

$$\mathrm{d}X_i(t) = -X_i(t)\mathrm{d}t + \mathrm{d}B_i(t) \quad 1 \leq i \leq k$$

- $X_i(t) \overset{d}{=} x_i e^{-t} + \frac{1}{\sqrt{2}} e^{-t} W(e^{2t} - 1)$ $\quad W$ is a standard B.M.
- $X(t)$ is Markovian, Gaussian, asymptotically stationary.
- invariant measure: $\mu(\mathrm{d}x) \sim N(0, I/2)$.

Poisson potential

Define the Poisson potential $V(\cdot)$:

$$V(x) = \int_{\mathbb{R}^d} K(x-y)\,\omega(\mathrm{d}y),$$

where

- deterministic shape function $K(x) \geq 0$ is continuous & compactly supported
- $\omega(\cdot)$ is a Poisson random measure with intensity measure $\lambda\,\mathrm{d}x$:
  - $\omega(\emptyset) = 0$
  - For any $\{A_i\} \subset R^d$ pairwise disjoint,

$$\omega\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \omega(A_i) \quad a.s.$$

  - For any $A \in \mathcal{B}(\mathbb{R}^d)$, $\omega(A)$ ~Poisson($\lambda\,|A|$).

Main Result

Theorem (X.)

For a $d-$dimensional O-U process , with probability one (w.r.t. $\mathbb{P}$)

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \exp\left\{\pm \int_0^t V(X(s))\, \mathrm{d}s\right\} = \pm\lambda_{\pm},$$

where λ_+ and λ_- are random variables taking values in $(0,\infty)$.

Remark

- *exponential moments have e^{ct} growth/decay speed*
- *however, the rate is highly influenced by the media*

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Compare to B.M.

- Positive exponential moment (Carmona, Molchanov '95)

$$\lim_{t\to\infty} \frac{\log\log t}{t\log t} \log \mathbb{E}_0 \exp\left\{\int_0^t V(B(s))\,\mathrm{d}s\right\} = c_1.$$

- Negative exponential moment (Sznitman '93)

$$\lim_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log \mathbb{E}_0 \exp\left\{-\int_0^t V(B(s))\,\mathrm{d}s\right\} = -c_2.$$

Variational formula

The following large deviation result holds $\mathbb{P}$-a.s.:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \exp\left\{ \pm \int_0^t V(X(s))\, \mathrm{d}s \right\}$$
$$= - \inf_{f\in\mathcal{F}} \left\{ \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla f|^2 \mp V(x) f^2(x) \right) \phi(x)\, \mathrm{d}x \right\}$$
$$= - \frac{1}{2}\pi^{-d/2} \inf_{g\in\mathcal{E}} \left\{ \int_{\mathbb{R}^d} |\nabla g|^2 + \left(|x|^2 \mp 2V(x)\right) g^2\, \mathrm{d}x \right\} + \frac{d}{2}$$

where $\mathcal{E} = \left\{ f(x) e^{-\frac{|x|^2}{2}} : f \in \mathcal{P}(\mathbb{R}^d), ||f||_\mu = 1 \right\}$

Fact

$$|x|^2 \mp 2V(x) \approx \begin{cases} |x|^2 & \text{for large } x \\ V(x) & \text{for small } x \end{cases}$$

Local behavior of V determines $\lambda_\pm$.

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$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x\exp\left\{\pm\int_0^t V(X(s))\,\mathrm{d}s\right\}$$

$$=-\inf_{f\in\mathcal{F}}\left\{\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla f|^2\mp V(x)f^2(x)\right)\phi(x)\,\mathrm{d}x\right\}$$

$$=-\frac{1}{2}\pi^{-d/2}\inf_{g\in\mathcal{E}}\left\{\int_{\mathbb{R}^d}|\nabla g|^2+\left(|x|^2\mp 2V(x)\right)g^2\,\mathrm{d}x\right\}+\frac{d}{2}$$

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$$|x|^2\mp 2V(x)\approx\begin{cases}|x|^2 & \text{for large } x\\ V(x) & \text{for small } x\end{cases}$$

Local behavior of V determines $\lambda_\pm$.

Remarks & Future plan

Remark

- If stationary $|V(x)| << |x|^2$, similar LDP result holds.

Future plan

- $V(\cdot)$ be general stationary potential?
- Annealed case? i.e. large time behavior of

$$\mathbb{E} \times \mathbb{E}_x \exp\left\{ \pm \int_0^\infty V(X(s))\, \mathrm{d}s \right\}$$

- Non-stationary potential?
- Tail/small value estimate of $\lambda_\pm$.

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Thank You!