Quenched Asymptotic for Ornstein-Uhlenbeck process of Poisson Potential

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Random Motion in Random Media

- **Random motion**: $\mathbb{R}^d$ valued Markovian process $X(t)$
  
  - $P_x$, $E_x$

- **Random media**: stationary random potential $V(\cdot)$
  
  - $P$, $E$

- $X(t)$ and $V(\cdot)$ are independent.

- Consider

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u_{\pm}(t, x) \overset{\text{def}}{=} E_x \exp \left\{ \pm \int_0^t V(X(s)) \, ds \right\}
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Q: How do $u_+$ and $u_-$ behave for $t$ large?
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Q: How do $u_+$ and $u_-$ behave for $t$ large?
Why exponential moment?

- \( u(t, x) \) is the normalizing constant of the random Gibbs measure \( \mu_{t, \omega} \)

\[
\frac{d\mu_{t, \omega}}{dP_x} = \frac{1}{u(t, x)} \exp \left\{ \pm \int_0^t V(X_s) \, ds \right\}
\]

- (Feynman-Kac) \( u(t, x) \) solves

\[
\begin{align*}
\partial_t u(t, x) &= \mathcal{L} u(t, x) \pm V(x) u(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d, \\
u(0, x) &= 1, \quad x \in \mathbb{R}^d,
\end{align*}
\]

\( \mathcal{L} \) – infinitesimal generator of the Markov operator \( T_t \)

\[
T_t(f)(x) \overset{\text{def}}{=} \mathbb{E}_x f(X(t)).
\]
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$L$ — infinitesimal generator of the Markov operator $T_t$

$$T_t(f)(x) \overset{def}{=} \mathbb{E}_x f(X(t)).$$
Brownian motion under stationary potentials have been well studied:

- Sznitman ’93
- Carmona, Molchanov ’95
- Gartner, König, Molchanov ’00
- Gartner, König ’05
- Chen ’11
Our Focus

\( X(t) \) be **Ornstein-Uhlenbeck (O-U)** process.

Why O-U?

- Extensive applications for *stationary* dynamics in other fields:
  - physical science: noisy relaxation process
  - finance: interest rate derivatives
  - biochemistry: model peptide bond angle of water molecules

- require a different strategy other than B.M case.
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Let $X(t)$ be a $d$-dim O-U process $X(t)$ starting at $x = (x_1, \ldots, x_d)$:

$$dX_i(t) = -X_i(t)dt + dB_i(t) \quad 1 \leq i \leq k$$

- $X_i(t) \overset{d}{=} x_i e^{-t} + \frac{1}{\sqrt{2}} e^{-t} W(e^{2t} - 1) \quad W$ is a standard B.M.
- $X(t)$ is Markovian, Gaussian, asymptotically stationary.
- invariant measure: $\mu(dx) \sim N(0, I/2)$. 
Define the **Poisson potential** \( V(\cdot) \):

\[
V(x) = \int_{\mathbb{R}^d} K(x - y) \omega(dy),
\]

where

- deterministic **shape function** \( K(x) \geq 0 \) is continuous & compactly supported
- \( \omega(\cdot) \) is a **Poisson random measure** with intensity measure \( \lambda \, dx \):
  - \( \omega(\emptyset) = 0 \)
  - For any \( \{A_i\} \subset \mathbb{R}^d \) pairwise disjoint,
    \[
    \omega\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \omega(A_i) \quad a.s.
    \]
  - For any \( A \in \mathcal{B}(\mathbb{R}^d) \), \( \omega(A) \sim \text{Poisson}(\lambda |A|) \).
Main Result

Theorem (X.)

For a $d$–dimensional O-U process, with probability one (w.r.t. $P$)

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_x \exp \left\{ \pm \int_0^t V(X(s)) \, ds \right\} = \pm \lambda_{\pm},$$

where $\lambda_+$ and $\lambda_-$ are random variables taking values in $(0, \infty)$.

Remark

- exponential moments have $e^{ct}$ growth/decay speed
- however, the rate is highly influenced by the media
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Compare to B.M.

- Positive exponential moment (Carmona, Molchanov ’95)
  \[
  \lim_{t \to \infty} \frac{\log \log t}{t \log t} \log \mathbb{E}_0 \exp \left\{ \int_0^t V(B(s)) \, ds \right\} = c_1.
  \]

- Negative exponential moment (Sznitman ’93)
  \[
  \lim_{t \to \infty} \frac{(\log t)^{2/d}}{t} \log \mathbb{E}_0 \exp \left\{ -\int_0^t V(B(s)) \, ds \right\} = -c_2.
  \]
The following large deviation result holds P-a.s.:

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_x \exp \left\{ \pm \int_0^t V(X(s)) \, ds \right\}$$

$$= - \inf_{f \in \mathcal{F}} \left\{ \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla f|^2 \mp V(x)f^2(x) \right) \phi(x) \, dx \right\}$$

$$= - \frac{1}{2} \pi^{-d/2} \inf_{g \in \mathcal{E}} \left\{ \int_{\mathbb{R}^d} |\nabla g|^2 + (|x|^2 \mp 2V(x)) g^2 \, dx \right\} + \frac{d}{2}$$

where \(\mathcal{E} = \left\{ f(x)e^{-\frac{|x|^2}{2}} : f \in \mathcal{P}(\mathbb{R}^d), ||f||_{\mu} = 1 \right\}\)

Fact

$$|x|^2 \mp 2V(x) \approx \begin{cases} |x|^2 & \text{for large } x \\ V(x) & \text{for small } x \end{cases}$$

Local behavior of \(V\) determines \(\lambda_{\pm}\).
Variational formula

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Fact

$$|x|^2 \mp 2V(x) \approx \begin{cases} |x|^2 & \text{for large } x \\ V(x) & \text{for small } x \end{cases}$$

Local behavior of $V$ determines $\lambda_\pm$. 
The following large deviation result holds P-a.s.:

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_x \exp \left\{ \pm \int_0^t V(X(s)) \, ds \right\} = -\inf_{f \in \mathcal{F}} \left\{ \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla f|^2 \mp V(x)f^2(x) \right) \phi(x) \, dx \right\} = -\frac{1}{2} \pi^{-d/2} \inf_{g \in \mathcal{E}} \left\{ \int_{\mathbb{R}^d} |\nabla g|^2 + (|x|^2 \mp 2V(x)) g^2 \, dx \right\} + \frac{d}{2}
\]

where \( \mathcal{E} = \left\{ f(x)e^{-\frac{|x|^2}{2}} : f \in \mathcal{P}(\mathbb{R}^d), ||f||_\mu = 1 \right\} \)

Fact

\(|x|^2 \mp 2V(x) \approx \begin{cases} |x|^2 & \text{for large } x \\ V(x) & \text{for small } x \end{cases} \]

Local behavior of \( V \) determines \( \lambda_{\pm} \).
Remarks & Future plan

Remark

- If stationary $|V(x)| \ll |x|^2$, similar LDP result holds.

Future plan

- $V(\cdot)$ be general stationary potential?

- Annealed case? i.e. large time behavior of

  $$
  \mathbb{E} \times \mathbb{E}_x \exp \left\{ \pm \int_0^\infty V(X(s)) \, ds \right\}
  $$

- Non-stationary potential?

- Tail/small value estimate of $\lambda_{\pm}$.
References


Xing, F. *Almost sure asymptotic for Ornstein-Uhlenbeck processes of Poisson potential*, 2012, (submitted)
Thank You!