Small Ball Properties and Fractal Properties of Gaussian Random Fields

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Outline

- An introduction to fractal geometry
  - Hausdorff measure and Hausdorff dimension
  - Packing measure and packing dimension
- Exact Hausdorff measure functions for the range of fBm
- Exact packing measure functions for the range of fBm
- Chung's LIL for fBm and its exceptional sets

1. Introduction to fractal geometry

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. It generates many random sets, for example,

- Range $\quad X\left([0,1]^N\right) = \left\{X(t) :\ t \in [0,1]^N\right\}$
- Graph $\quad \mathrm{Gr}X\left([0,1]^N\right) = \left\{(t, X(t)) :\ t \in [0,1]^N\right\}$
- Level set $\quad X^{-1}(x) = \left\{t \in \mathbb{R}^N : X(t) = x\right\}$
- Excursion set $X^{-1}(F) = \left\{t \in \mathbb{R}^N : X(t) \in F\right\}$, $\forall F \subseteq \mathbb{R}^d$,
- The set of self-intersections, ...

In order to study them, we need some tools such as Hausdorff dimension and packing dimension from fractal geometry.

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In order to study them, we need some tools such as Hausdorff dimension and packing dimension from fractal geometry.

1.1 Definitions of Hausdorff measure and dimension

Let Φ be the class of functions $\varphi : (0, \delta) \to (0, \infty)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ such that

$$\frac{\varphi(2s)}{\varphi(s)} \le K \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

A function φ in Φ is often called a *measure function* or *gauge function*.

For example, $\varphi(s) = s^\alpha$ $(\alpha > 0)$ and $\varphi(s) = s^\alpha \log\log(1/s)$ are measure functions.

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Given $\varphi \in \Phi$, the *φ-Hausdorff measure* of $E \subseteq \mathbb{R}^d$ is defined by

$$\varphi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \varepsilon \right\}, \tag{1}$$

where $B(x, r)$ denotes the open ball of radius r centered at x. The sequence of balls satisfying the two conditions on the right-hand side of (1) is called an ε-covering of E.

It can be shown that φ-m is a metric outer measure and all Borel sets in $\mathbb{R}^d$ is φ-m measurable.

A function $\varphi \in \Phi$ is called *an exact Hausdorff measure function for E* if $0 < \varphi\text{-}m(E) < \infty$.

If $\varphi(s) = s^\alpha$, we write φ-$m(E)$ as $\mathcal{H}_\alpha(E)$.
The *Hausdorff dimension* of E is defined by

$$\begin{aligned}\dim_{\mathrm{H}} E &= \inf\left\{\alpha > 0 : \ \mathcal{H}_\alpha(E) = 0\right\} \\ &= \sup\left\{\alpha > 0 : \ \mathcal{H}_\alpha(E) = \infty\right\},\end{aligned}$$

Convention: $\sup \varnothing := 0$.

Lemma 1.1

1. $E \subseteq F \subseteq \mathbb{R}^d \ \Rightarrow \dim_{\mathrm{H}} E \le \dim_{\mathrm{H}} F \le d$.
2. (σ-stability):

$$\dim_{\mathrm{H}}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sup_{j\ge 1} \dim_{\mathrm{H}} E_j.$$

An upper density theorem

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *upper φ-density of μ at $x \in \mathbb{R}^d$* is defined as

$$\overline{D}_{\mu}^{\varphi}(x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{\varphi(2r)}.$$

Theorem 1.2 (Rogers and Taylor, 1961)

Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have

$$K^{-1}\mu(E) \inf_{x \in E} \left\{\overline{D}_{\mu}^{\varphi}(x)\right\}^{-1} \le \varphi\text{-}m(E) \le K\|\mu\| \sup_{x \in E} \left\{\overline{D}_{\mu}^{\varphi}(x)\right\}^{-1}.$$

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1.2 Packing measure and packing dimension

They were introduced by Tricot (1982), Taylor and Tricot (1985). For any $\varphi \in \Phi$ and $E \subseteq \mathbb{R}^d$, define

$$\varphi\text{-}P(E) = \lim_{\varepsilon\to 0} \sup \left\{ \sum_i \varphi(2r_i) : \{\overline{B}(x_i, r_i)\} \text{ is an } \varepsilon\text{-packing} \right\}$$

Here ε-packing means that the balls are disjoint, $x_i \in E$ and $r_i \leq \varepsilon$.

The packing measure φ-p of E is defined as:

$$\varphi\text{-}p(E) = \inf \left\{ \sum_n \varphi\text{-}P(E_n) : \ E \subseteq \bigcup_n E_n \right\}.$$

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If $\varphi(s) = s^{\alpha}$, we write φ-$p(E)$ as $\mathcal{P}_{\alpha}(E)$. The packing dimension of E is defined as:

$$\dim_{\mathrm{P}} E = \inf\{\alpha > 0 : \ \mathcal{P}_{\alpha}(E) = 0\}.$$

Comparison between $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$:
For any $\varphi \in \Phi$ and $E \subseteq \mathbb{R}^d$,

$$\varphi\text{-}m(E) \le \varphi\text{-}p(E), \qquad \dim_{\mathrm{H}} E \le \dim_{\mathrm{P}} E.$$

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A lower density theorem

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *lower φ-density* of μ at $x \in \mathbb{R}^d$ is defined as

$$\underline{D}_\mu^\varphi(x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{\varphi(2r)}.$$

Theorem 1.3 (Taylor and Tricot, 1985)

Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have

$$K^{-1}\mu(E) \inf_{x \in E} \left\{ \underline{D}_\mu^\varphi(x) \right\}^{-1} \le \varphi\text{-}p(E) \le K\|\mu\| \sup_{x \in E} \left\{ \underline{D}_\mu^\varphi(x) \right\}^{-1}.$$

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Example: Cantor's set

Let C denote the standard ternary Cantor set in $[0\,,1]$. At the nth stage of its construction, C is covered by 2^n intervals of length/diameter 3^{-n} each.
It can be proved that

$$\dim_{_\mathrm{H}} C = \dim_{_\mathrm{P}} C = \log_3 2.$$

By using the upper and lower density theorems, one can prove that

$$0 < \mathcal{H}_{\log_3 2}(C) \le \mathcal{P}_{\log_3 2}(C) < \infty.$$

Example: the range of Brownian motion

Let $B([0, 1])$ be the image of Brownian motion in $\mathbb{R}^d$. Lévy (1948) and Taylor (1953) proved that

$$\dim_{\mathrm{H}} B([0, 1]) = \min\{d, 2\} \qquad \text{a.s.}$$

Ciesielski and Taylor (1962), Ray and Taylor (1964) proved that

$$0 < \varphi_d\text{-}m\big(B([0, 1])\big) < \infty \quad \text{a.s.,}$$

where

$$\begin{aligned} \varphi_1(r) &= r, \\ \varphi_2(r) &= r^2 \log(1/r) \log\log\log(1/r), \\ \varphi_d(r) &= r^2 \log\log(1/r), \qquad \text{if } d \geq 3. \end{aligned}$$

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Taylor and Tricot (1985) proved that

$$\dim_{\mathrm{P}} B([0,1]) = \min\{d, 2\}$$

and, if $d \geq 3$, then

$$0 < \psi\text{-}p\big(B([0,1])\big) < \infty \quad \text{a.s.},$$

where $\psi(r) = r^2/\log\log(1/r)$.

LeGall and Taylor (1986) proved that, if $d = 2$, then for any measure function φ, either $\varphi\text{-}p\big(B([0,1])\big) = 0$ or ∞.

Question: How to extend the above results to Gaussian random fields? (Consider fractional Brownian motion only.)

2. Fractional Brownian motion

For $H \in (0,1)$, the fBm $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ with index H is a centered (N, d)-Gaussian field whose covariance function is

$$\mathbb{E}\left[B_i^H(s) B_j^H(t)\right] = \frac{1}{2}\,\delta_{ij}\ \left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right),$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

- When $N = 1$ and $H = 1/2$, B^H is Brownian motion.
- B^H is H-self-similar and has stationary increments.

Kahane (1985) proved that

$$\dim_{\rm H} B^H([0,1]^N) = \min\left\{d,\ \frac{N}{H}\right\} \qquad \text{a.s.}$$

2.1 Exact Hausdorff measure function for $B^H([0,1]^N)$

Theorem 2.1 (Talagrand, 1995, 1998)

Let $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$.

(i). If $N < Hd$, then

$$K^{-1} \le \varphi_1\text{-}m\big(B^H([0,1]^N)\big) \le K, \qquad \text{a.s.}$$

where $\varphi_1(r) = r^{\frac{N}{H}} \log\log(1/r)$.

(ii). If $N = Hd$, then $\varphi_2\text{-}m\big(B^H([0,1]^N)\big)$ is σ-finite, where

$$\varphi_2(r) = r^d \log(1/r) \log\log\log(1/r).$$

Proof of the upper bound

An economic covering of $B^H([0,1]^N)$ must reflect the local oscillation behavior of the sample paths of B^H. Talagrand (1995) classified the points in $[0,1]^N$ into "good" points and "bad" points according to the local asymptotic behavior of fBm at these points.

Typically, $t_0 \in [0,1]^N$ is "good" if the oscillation of B^H around $B^H(t_0)$ is small on a sequence of balls $U(t_0, r_n)$, where $r_n \downarrow 0$, so that $B^H(U(t_0, r_n))$ can be covered by balls with small radius. This is precisely characterized by Chung's LIL for fBm at t_0.

At "bad" points, the the oscillation can be as large as given by the uniform modulus of continuity. Fortunately, there are not many such points.

Here is Talagrand's key estimate.

Proposition 2.2 (Talagrand, 1995)

There exists a constant $\delta_1 > 0$ such that for any $0 < r_0 \le \delta_1$, we have

$$\mathbb{P}\bigg\{ \exists r \in [r_0^2, r_0] \ \text{ such that} \qquad \max_{|t|\le r} |B^H(t)| \le K\, r^H \big(\log\log(1/r)\big)^{-H/N} \bigg\} \ge 1 - \exp\bigg(-\Big(\log \frac{1}{r_0}\Big)^{\frac{1}{2}}\bigg).$$

For $k \geq 1$, consider the set

$$R_k = \Big\{ t \in [0,1]^N : \exists r \in [2^{-2k}, 2^{-k}] \text{ such that}$$
$$\sup_{|s-t| \leq r} |B^H(s) - B^H(t)| \leq K r^H \big(\log\log(1/r)\big)^{-H/N} \Big\}.$$

Since B^H has stationary increments, Proposition 2.2 and Fubini's theorem imply that almost surely

$$L_N(R_k) \geq 1 - \exp(-\sqrt{k/4}) \text{ infinitely often.}$$

This leads to an upper bound for φ_1-$m\big(B^H([0,1]^N)\big)$.

The proof of the lower bound makes use of the upper density theorem and strong local nondeterminism. This part is omitted.

Further results

The problems on the exact Hausdorff measure functions for the graph set and level set of B^H were studied by Xiao (1997, 1998).

Extensions of Talagrand (1995) to anisotropic Gaussian random fields with stationary increments are done in Luan and Xiao (2012).

Research problems:

- For fBm, the problem for the exact Hausdorff measure function in the critical case of $N = Hd$ is open.
- All problems for fractional Brownian sheets are open.

2.2. Exact packing measure function for $B^H([0,1]^N)$

Theorem 2.3 (Xiao, 1996, 2003)

Let $B^H = \{B^H(t),\, t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$. If $N < Hd$, then there exists a finite constant $K \geq 1$ such that

$$K^{-1} \leq \varphi_3\text{-}p(B^H([0,1]^N)) \leq K, \quad \text{a.s.}$$

where $\varphi_3(r) = r^{\frac{N}{H}}\big(\log\log(1/r)\big)^{-N/(2H)}$.

For proving Theorem 2.3, we consider the sojourn measure

$$T(r) = \int_{\mathbb{R}^N} \mathcal{I}_{\{B^H(t)|\le r\}} dt.$$

This is a nonnegative, non-decreasing and self-similar process.

A key ingredient is the following small ball probability estimate for $T(1)$.

Proposition 2.4 (Xiao, 1996, 2003)

Assume that $N < Hd$. Then there exists a positive and finite constant $K \ge 1$, depending only on H, N and d such that for any $0 < \varepsilon < 1$,

$$\exp\left(-\frac{K}{\varepsilon^{2H/N}}\right) \le \mathbb{P}\{T(1) < \varepsilon\} \le \exp\left(-\frac{1}{K\varepsilon^{2H/N}}\right).$$

This leads to the following Chung's LIL for $T(r)$.

Theorem 2.4 (Xiao, 1996, 2003)

If $N < Hd$, then with probability one,

$$\liminf_{r \to 0} \frac{T(r)}{\varphi_3(r)} = K, \tag{2}$$

where $0 < K < \infty$ is a constant depending on H, N and d only.

By the stationarity of increments of B^H and the lower density theorem, we derive the lower bound in Theorem 2.3.

The proof of upper bound in Theorem 2.3 requires a different argument.

Research problems

- In light of Proposition 2.4, we conjecture that

$$\lim_{\varepsilon \to 0} \varepsilon^{2H/N} \log \mathbb{P}\{T(1) < \varepsilon\} \ \text{exists.}$$

- For fBm, the problems for the exact packing measure functions for the graph, level sets and other sets are all open.
- All problems for the Brownian sheet, fractional Brownian sheets and other anisotropic Gaussian random fields are all open.

2.3. Chung's LIL and its exceptional sets

Using small ball probability, one proves the following Chung'
LIL.

Theorem 2.5 (Monrad and Rootzen, 1995; Li and Shao, 2001; Xiao, 1997)

For any $t \in \mathbb{R}^N$,

$$\liminf_{r \to 0} \sup_{|s| \le r} \frac{|B^H(t+s) - B^H(t)|}{r^H/(\log\log 1/r)^{H/N}} = K \qquad \text{a.s.},$$

where K is a positive and finite constant.

When $N = 1$, $K = \kappa$, which is the small ball constant in Li and Linde (1998).

In the case of $N = 1$, we use the SLND to prove the following result on the modulus of non-differentiability of B^H.

Theorem 2.6 (Hwang, Wang and Xiao, 2012)

With probability 1,

$$\liminf_{r \to 0} \inf_{t \in [0,1]} \frac{\sup_{|s| \le r} |B^H(t+s) - B^H(t)|}{\big(r/\log 1/r\big)^H} = \kappa.$$

Hence, for any constant $\gamma \ge 1$, the random set

$$S(\gamma) = \left\{ t \in [0,1] : \liminf_{r \to 0} \frac{\sup\limits_{|s| \le r} |B^H(t+s) - B^H(t)|}{\big(r/\log 1/r\big)^H} \le \gamma\kappa \right\}$$

has Lebesgue measure 0.

By applying the general results in Khoshnevisan, Peres and Xiao (2000) on limsup random fractals and small ball probability estimates, we prove

Theorem 2.7 (Hwang, Wang and Xiao, 2012)

For any $\gamma \ge 1$,

$$\dim_{\mathrm{H}} S(\gamma) = 1 - \gamma^{-1/H} \quad \text{and} \quad \dim_{\mathrm{P}} S(\gamma) = 1 \qquad \text{a.s.}$$

Moreover, for any fixed set $E \subseteq [0, 1]$,

$$\mathbb{P}\{S(\gamma) \cap E \neq \emptyset\} = \begin{cases} 1, & \text{if } \dim_{\mathrm{P}} E > \gamma^{-1/H}, \\ 0, & \text{if } \dim_{\mathrm{P}} E < \gamma^{-1/H}. \end{cases}$$

Thank you