Thinning Invariant Sequences

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Small Deviation Probabilities: Theory and Applications

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Advertisement: spin glasses

Sherrington-Kirkpatrick model:

For each $N \in \mathbb{N}$, $H_N : \{+1,-1\}^N \to \mathbb{R}$ is a random function

$$H_N(\sigma) = \sum_{i,j=1}^{N} \frac{J_{ij}}{\sqrt{2N}} \sigma_i \sigma_j$$

where $\sigma = (\sigma_1, \dots, \sigma_N)$ and $(J_{ij})_{i,j=1}^N$ are i.i.d., $\mathcal{N}(0,1)$.

Two key ideas.

- Gaussian inequalities: Slepian's lemma.
- Exchangeability.

1/10 Bernoulli-p thinning

Let $\mathscr{X} \stackrel{\text{def}}{:=} \{0,1\}$. Let $p \in (0,1]$.

- Suppose

$$(X_1, X_2, \dots) \sim \mu \in \mathcal{M}(\mathscr{X}^{\mathbb{N}})$$

- Independently,

$$(B_1, B_2, \dots) \sim \text{ i.i.d. Bernoulli}(p)$$

- Let $K_1 < K_2 < \dots \ =$ indices k s.t. $B_k = 1$
- $\theta_p(\mu) \in \mathcal{M}(\mathscr{X}^{\mathbb{N}}) =$ marginal distribution $(X_{K_1}, X_{K_2}, \dots)$.

$$\begin{array}{ccccccc}
 & X_1 & X_2 & X_3 & X_4 & X_5 & \cdots \\
B_k\colon & 1 & 1 & 0 & 1 & 0 & \cdots \\
 & \downarrow & \downarrow & & \swarrow & & \\
 & X_{K_1} & X_{K_2} & X_{K_3} & \cdots & &
\end{array}$$

2/10 Thinning invariance

$$
\begin{array}{lcccccc}
 & X_1 & X_2 & X_3 & X_4 & X_5 & \cdots \\
B_k: & 1 & 1 & 0 & 1 & 0 & \cdots \\
 & \downarrow & \downarrow & & \swarrow & & \\
 & Y_1 & Y_2 & Y_3 & \cdots & &
\end{array}
$$

Definition: *Thinning invariance*:

$$\theta_p(\mu) = \mu \quad \text{for all } p \in (0,1].$$

<u>Example:</u> Let $X_1, X_2, \dots$ be i.i.d.

<u>Example 2.</u> Let $X_1, X_2, \dots$ be conditionally i.i.d.

E.g. $U \sim \mathrm{Unif}([0,1])$.

Conditional on U: $X_1, X_2, \dots \sim$ i.i.d, Bernoulli(U).

Thinning invariant $\neq$ spreadable/exchangeable

Definition: *Spreadability*:

- $\forall$ non-random $k_1 < k_2 < \dots$,
- given $(X_1, X_2, \dots) \sim \mu$,

$$(X_{k_1}, X_{k_2}, \dots) \sim \mu\,.$$

Ryll-Nardzewski showed that spreadability equals exchangeability.

Definition: *Exchangeability*

- $\forall$ "finite" permutation $\pi : \mathbb{N} \to \mathbb{N}$,
- given $(X_1, X_2, \dots) \sim \mu$,

$$(X_{\pi(1)}, X_{\pi(2)}, \dots) \sim \mu\,.$$

References: for thinning invariant point processes.

Olav Kallenberg, "Random measures."
Mathes, Kerstan and Mecke, "Infinitely divisible point processes."

Reminder: De Finetti's theorem & random measures

$\mathcal{M}(\mathscr{X}) =$ all Borel measures μ on $\mathscr{X}$.

$\mathcal{M}(\mathcal{M}(\mathscr{X})) =$ all Borel measures Q on $\mathcal{M}(\mathscr{X})$.

For $Q \in \mathcal{M}(\mathcal{M}(\mathscr{X}))$:

- $\mu \sim Q$
- conditional on μ: $(X_1, X_2, \dots) \sim$ i.i.d., μ
- $\mu_Q =$ marginal distribution of $(X_1, X_2, \dots)$

De Finetti's Theorem:
$\{\mu \in \mathcal{M}(\mathscr{X}^{\mathbb{N}}) \,:\, \text{exchangeable}\} = \{\mu_Q \,:\, Q \in \mathcal{M}(\mathcal{M}(\mathscr{X}))\}$

5/10 Counterexamples to exchangeability

1. Choose $0 < \xi_1 < \xi_2 < \dots \sim$ Poisson point process.
Independently, choose $U \sim \mathrm{Unif}([0,2))$.

$$X_n = \mathbf{1}_{2\mathbb{Z}+[0,1)}(\ln \xi_n + U) \quad \text{for } n \in \mathbb{N},$$

2. Choose $0 < \xi_1 < \xi_2 < \dots \sim$ Poisson point process.
Independently, let $W : \mathbb{R} \to [0,1] \sim$ reflected Brownian motion.

$$U_n = W(\ln \xi_n) \in [0,1] \quad \text{for each } n \in \mathbb{N}.$$

Conditional on $(U_1, U_2, \dots)$: $X_1, X_2, \dots$ independent

$$X_n \sim \mathrm{Bernoulli}(Y_n)\,.$$

- Note $\{(\ln \xi_n, X_n)\}_{n\in\mathbb{N}}$ is a mixed Poisson process on $\mathbb{R} \times \mathscr{X}$.

Characterization theorem

- Let $\mathcal{M}_{\mathrm{Leb}}(\mathbb{R} \times \mathscr{X})$ denote the set of all Borel measures ρ on $\mathbb{R} \times \mathscr{X}$ such that $\rho(\cdot \times \mathscr{X}) =$ Lebesgue measure.
- Let $\mathcal{M}_{\mathrm{I}}(\mathcal{M}_{\mathrm{Leb}}(\mathbb{R} \times \mathscr{X}))$ denote the set of all Borel probability measures Q on $\mathcal{M}_{\mathrm{Leb}}(\mathbb{R} \times \mathscr{X})$ such that

$$Q(\{\rho : \rho \circ \tau_t^{-1} \in \cdot\}) = Q,$$

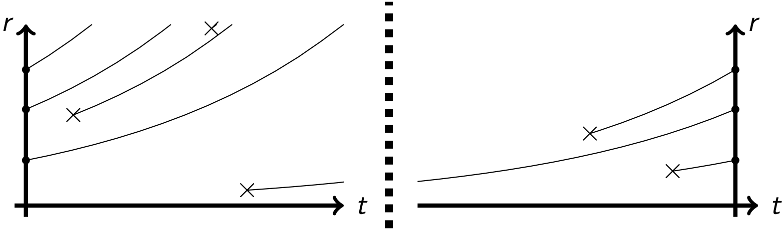
for all $t \in \mathbb{R}$, where $\tau_t(s,x) = (s+t,x)$.

Theorem [Wei, S] $\{\mu \in \mathcal{M}(\mathscr{X}^{\mathbb{N}}) : \forall p \in (0,1], \theta_p(\mu) = \mu\}$ *is isomorphic as a simplex to* $\mathcal{M}_{\mathrm{I}}(\mathcal{M}_{\mathrm{Leb}}(\mathbb{R} \times \mathscr{X}))$.

- Given Q, let $\rho \in \mathcal{M}_{\mathrm{Leb}}(\mathbb{R} \times \mathscr{X})$ be chosen according to Q.
- Let ρ' be a new measure: $\rho'(dt \times dx) = e^t \rho(dt \times dx)$.
- Let Ξ be a Poisson process on $\mathbb{R} \times \mathscr{X}$ with intensity ρ'.
- A.s., $\Xi = \sum_{n=1}^{\infty} \delta_{(\xi_n, X_n)}$ with $\xi_1 < \xi_2 < \dots$.
- $\mu_Q \in \mathcal{M}(\mathscr{X}^{\mathbb{N}}) =$ the marginal distribution of $(X_1, X_2, \dots)$.

Key idea: Hoyle's super-creationist model

In one of his papers, Aldous describes the following as Hoyle's "steady state model."



- At any time, standard Poisson point process.
- Space dilates at a constant rate.
- New points added according to space-time Poisson process.

Relation to thinning invariant partition structure

A random partition structure "is" a random sequence $(\xi_1, \xi_2, \dots)$:

- $\xi_1 > \xi_2 > \cdots > 0$
- $\xi_1 + \xi_2 + \cdots = 1$

Thinning here has two steps:

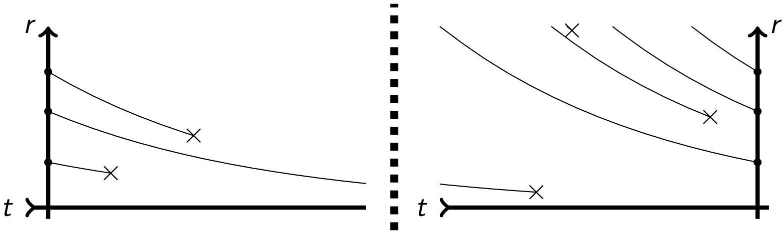
- $(\xi_{K_1}, \xi_{K_2}, \dots)$,
- $(\xi_{K_1}/Z\,,\ \xi_{K_2}/Z\,, \dots)$ where $Z = \xi_{K_1} + \xi_{K_2} + \dots$.

Example. For $0 < m < 1$, let $\xi_1 > \xi_2 > \dots \sim \mathrm{PPP}(mx^{-m-1}\,dx)$.
Let $Z = \xi_1 + \xi_2 + \dots$.
$(\xi_1/Z\,,\ \xi_2/Z\,,\ \dots) \sim$ Poisson-Dirichlet PD$(m, 0)$

<u>References:</u>

Pitman and Yor, The two parameter Poisson-Dirichlet process derived from a stable subordinator.
Pitman, Poisson-Kingman partitions.

Ergodic theorem



$$-t^{-1}\ln \xi_1(t) \to 1/m$$
$$\Rightarrow \xi_n(0) \sim n^{-1/m}$$

Idea: *still only heuristic*. Multiply $\xi_n(0)$ by $(e^t N_n(-t))^{1/m}$ and then claim the result is a thinning invariant sequence.

∗ This problem arises in the simplest spin glass, Derrida's REM. Aizenman and Ruzmaikina solved the cavity step dynamics. But $N \to N+1$ also *doubles* the configuration space.

10/10 Rockets in Huntsville

Thank you!