Small-ball probabilities for the volume of random convex sets

Peter Pivovarov (joint work with Grigoris Paouris)

Texas A&M University

June 6, 2012

Setting

- $\mathcal{P}_n$ - all Lebesgue absolutely continuous prob. measures μ on $\mathbb{R}^n$ s.t. $\left\| \frac{d\mu}{dx} \right\|_\infty = 1$
- $N \geqslant n$
- $\mu_1, \ldots, \mu_N \in \mathcal{P}_n$
- $X_1, \ldots, X_N$ independent random vectors: $X_i \sim \mu_i$

Absolute convex hull:

$$K_N := \operatorname{conv}\{\pm X_1, \ldots, \pm X_N\}$$

Zonotope = Minkowski sum of segments $[-X_i, X_i]$:

$$Z_N := \sum_{i=1}^{N} [-X_i, X_i] = \Big\{ \sum_{i=1}^{N} \lambda_i X_i : \lambda_i \in [-1, 1],\ i = 1, \ldots, N \Big\}.$$

Viewpoint

- Form the $n \times N$ random matrix

$$[X_1 \cdots X_N] : \mathbb{R}^N \to \mathbb{R}^n$$

- If $C \subset \mathbb{R}^N$ is a convex body, then

$$[X_1 \cdots X_N]C = \left\{ \sum_{i=1}^{N} c_i X_i : c = (c_i) \in C \right\}$$

- Convex hull: take $C = B_1^N$ (unit ball in ℓ_1^N - "cross-polytope")

$$K_N = [X_1 \cdots X_N] B_1^N$$

- Zonotope: take $C = B_\infty^N = [-1, 1]^N$

$$Z_N = [X_1 \cdots X_N] B_\infty^N$$

- D_n - Euclidean ball of volume one
- λ_{D_n} - volume restricted to D_n

Theorem (Paouris, P. '12)

Suppose $N \geqslant n$ and $\mu_1, \ldots, \mu_N \in \mathcal{P}_n$. Let $C \subset \mathbb{R}^N$ be a convex body and $p > 0$. Then

$$\mathbb{E}_{\otimes \mu_i} \mathrm{vol}_n ([X_1 \cdots X_N] C)^p \geqslant \mathbb{E}_{\otimes \lambda_{D_n}} \mathrm{vol}_n ([X_1 \cdots X_N] C)^p .$$

- $C = B_1^N$ get K_N ([Groemer, '76])
- $C = B_\infty^N$ get Z_N ([Bourgain, Meyer, Milman, Pajor, '88])

Theorem

Suppose $N \geqslant n$ and $\mu_1, \ldots, \mu_N \in \mathcal{P}_n$. Let $C \subset \mathbb{R}^N$ be a 1-unconditional convex body. Then for each $\alpha > 0$,

$$\begin{aligned}
&\mathbb{P}_{\otimes \mu_i}\left(\mathrm{vol}_n\left([X_1 \cdots X_N]C\right) \leqslant \alpha\right) \\
&\qquad \leqslant \mathbb{P}_{\otimes \lambda_{D_n}}\left(\mathrm{vol}_n\left([X_1 \cdots X_N]C\right) \leqslant \alpha\right).
\end{aligned}$$

Key ingredients: [Brascamp, Lieb, Luttinger '74], [Groemer, '76].

Proposition

If $n \leqslant N \leqslant e^n$, then

$$\mathbb{E}_{\otimes\lambda_{D_n}} \mathrm{vol}_n\left(K_N\right)^{\frac{1}{n}} \simeq \sqrt{\frac{\log(2N/n)}{n}}.$$

For any $N \geqslant n$,

$$\mathbb{E}_{\otimes\lambda_{D_n}} \mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}} \simeq \frac{N}{\sqrt{n}}.$$

- Lower bounds for K_N: [Gluskin '88], [Giannopoulos, Tsolomitis '03].
- Upper bounds for max. vol. $\mathrm{conv}\{\pm X_i\}$: [Carl, Pajor, '87], [Barany, Furedi, '88], [Gluskin, '88], [Ball, Pajor, '90].
- For Z_N: [Bourgain, Meyer, Milman, Meyer '88]

Small deviations

- D_n - Euclidean ball of volume one
- λ_{D_n} - volume restricted to D_n

Main goal: Find the dependence on n, N and ε in

$$\mathbb{P}_{\otimes\lambda_{D_n}}\left(\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant \varepsilon\sqrt{\frac{\log(2N/n)}{n}}\right)$$

and

$$\mathbb{P}_{\otimes\lambda_{D_n}}\left(\mathrm{vol}_n\left(Z_N\right)^{1/n} \leqslant \frac{\varepsilon N}{\sqrt{n}}\right).$$

Theorem (Litvak, Pajor, Rudelson, Tomczak-Jaegermann, '05)

Assume

- *coordinates of each* X_i *are iid, symmetric sub-Gaussian;*
- $(1+\zeta)n \leqslant N \leqslant e^n$, *where* $\zeta > 1/\ln n$, *and* $\beta \in (0, 1/2)$*;*

Then

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n(K_N)^{1/n} \leqslant c(\zeta)\sqrt{\frac{\beta\log(2N/n)}{n}}\right) \leqslant \exp(-c_1 N^{1-\beta} n^{\beta}).$$

Theorem (Paouris, P.)

Let $\mu_1, \ldots, \mu_N \in \mathcal{P}_n$. *Let* $\delta > 1$ *and* $\varepsilon \in (0,1)$. *If* $n \leqslant N \leqslant ne^{\delta^2}$, *then*

$$\mathbb{P}_{\otimes \mu_i}\left(\mathrm{vol}\,(K_N)^{1/n} \leqslant \frac{c_1\varepsilon}{\delta}\sqrt{\frac{\log(2N/n)}{n}}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}.$$

Moreover, if $n \leqslant N \leqslant e^n$, *then*

$$\mathbb{P}_{\otimes \mu_i}\left(\mathrm{vol}\,(K_N)^{1/n} \leqslant \frac{c_2\varepsilon}{\delta}\sqrt{\frac{\log(2N/n)}{n}}\right) \leqslant \varepsilon^{c_3 N^{1-1/\delta^2} n^{1/\delta^2}}.$$

Theorem (Paouris, P.)

Let $n \leqslant N \leqslant e^n$ and $\mu_1, \ldots, \mu_N \in \mathcal{P}_n$. Let $\varepsilon \in (0,1)$. Then

$$\mathbb{P}_{\otimes \mu_i}\left(\mathrm{vol}_n(Z_N)^{1/n} \leqslant \frac{c\varepsilon N}{\sqrt{n}}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}.$$

Asymptotically correct as $\varepsilon \to 0$.

Gaussian case

γ_n - standard Gaussian measure on $\mathbb{R}^n$

$$d\gamma_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx.$$

For $n \leqslant N \leqslant e^n$,

$$\mathbb{P}_{\otimes \lambda_{D_n}} \left(\mathrm{vol}_n \left([X_1 \cdots X_N] C \right)^{1/n} \leqslant \alpha \right) \approx \mathbb{P}_{\otimes \gamma_n} \left(\mathrm{vol}_n \left([X_1 \cdots X_N] C \right)^{1/n} \leqslant \alpha \right).$$

Estimate moments:

$$\mathbb{E}\, \mathrm{vol}_n \left(GC \right)^{-p}$$

where $G = (\gamma_{ij})$ is an $n \times N$ matrix with iid standard Gaussian entries and $p > 0$.

Proposition

Let $N \geqslant n$ and let $C \subset \mathbb{R}^N$ be a convex body. Let G be an $n \times N$ random matrix with iid standard Gaussian entries. Then for all $p > -(N-n+1)$,

$$\mathbb{E}\,\mathrm{vol}_n\,(GC)^p = \mathbb{E}\det\,(GG^*)^{\frac{p}{2}} \int_{G_{N,n}} \mathrm{vol}_n\,(P_E C)^p \, d\nu_{N,n}(E).$$

$p = 1$: Gaussian representation of intrinsic volumes [Tsirelson, '85]

$n = 1$, $C = -C$: Negative moments of norms -

- [Latala, Oleszkiewicz, '05]
- [Klartag, Vershynin, '07]

Goal: compute moments

$$\int_{G_{N,n}} \mathrm{vol}_n\,(P_E C)^{-p}\, d\nu_{N,n}(E)$$

for $p > 0$.

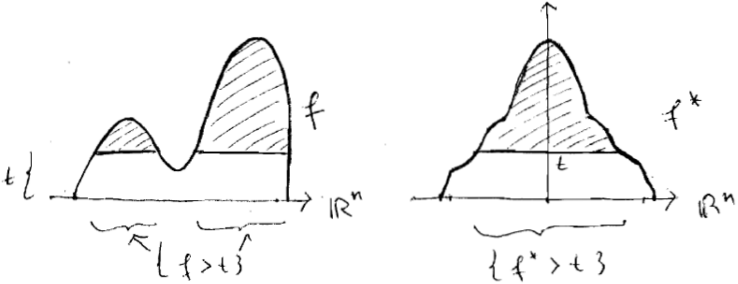
$C = B_\infty^N$	$C = B_1^N$
• Affine quermassintegrals	• $G_{N,n} \rightsquigarrow S^{n-1}$
• [Lutwak, '88]	• Alexsandrov-Fenchel
• [Grinberg, '91]	• negative moments for norms
• [Bourgain, Milman, '87]	

Symmetrization/Rearrangements

For an integrable $f : \mathbb{R}^n \to \mathbb{R}^+$, set

$$f^*(x) = \inf\{t > 0 : \mathrm{vol}_n\,(B(0,|x|)) \geqslant \mathrm{vol}_n\,(\{f > t\})\}.$$

- $f^*(x) = f^*(y)$ if $|x| = |y|$.
- $f^*(x) \geqslant f^*(y)$ if $|x| \leqslant |y|$.
- $\mathrm{vol}\,(\{f > t\}) = \mathrm{vol}\,(\{f^* > t\})$ for each t.



Theorem (Brascamp, Lieb & Luttinger, '74)

Let $f_1, \dots, f_M : \mathbb{R} \to \mathbb{R}^+$, $u_1, \dots, u_M \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{M} f_i(\langle x, u_i \rangle) dx \leqslant \int_{\mathbb{R}^n} \prod_{i=1}^{M} f_i^*(\langle x, u_i \rangle) dx$$

Corollary

Let $L = -L \subset \mathbb{R}^n$ be convex, $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}^+$. Then

$$\int_L \prod_{i=1}^{n} f_i(x_i) dx \leqslant \int_L \prod_{i=1}^{n} f_i^*(x_i) dx.$$

$$\int_L \prod_{i=1}^{n} f_i(x_i)dx \leqslant \int_L \prod_{i=1}^{n} f_i^*(x_i)dx.$$

- When $f_i = \mathbb{1}_{S(i)}$, $S(i) \subset \mathbb{R}$, compact: [Pfiefer, '90]
- Extensions for L non-convex: [Draghici, '06]