Strong Analytic Solutions of Fractional Cauchy Problems

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Introduction and Known Results

Cauchy problems

$$\frac{\partial u}{\partial t} = Lu$$

model diffusion processes and have appeared as an essential tool for the study of dynamics of various complex stochastic processes arising in anomalous diffusion in

- physics (Metzler, R. and Klafter, J. 2000, Zaslavsky and G. M. Choas 2000),
- finance (Gorenflo, R., Mainardi, F., Scalas, E. and Raberto, M 2001),
- hydrology (Benson, D. A., Wheatcraft, S.W., and Meerschaert, M. M. 2000), and
- cell biology (Saxton, M. J. and Jacobson, K. 1997). For example, experimental studies of the motion of macromolecules in a cell membrane show apparent sub diffusive motion with several simultaneous diffusive modes (Saxton, M. J. and Jacobson, K. 1997).

Introduction and Known Results

The simplest case $L = \Delta = \sum_j \partial^2/\partial x_j^2$ governs a Brownian $B(t)$ with density $u(t,x)$, for which the square root scaling

$$u(t,x) = t^{-1/2} u(1, t^{-1/2} x)$$

pertains (Einstein, A., 1906)

Introduction and Known Results

- The fractional Cauchy problem

$$\partial^\beta u/\partial t^\beta = Lu$$

with $0 < \beta < 1$ models anomalous sub-diffusion, in which a cloud of particles spreads slower than the square root of time.

- When $L = \Delta$, the solution $u(t, x)$ is the density of a time-changed Brownian motion $B(E(t))$, where the non-Markovian time change

$$E(t) = \inf\{\tau > 0; D(\tau) > t\}$$

is the inverse, or first passage time of a stable subordinator $D(t)$ with index β. The scaling $D(ct) = c^{1/\beta}D(t)$ in law implies $E(ct) = c^\beta E(t)$ in law for the inverse process, so that $u(t, x) = t^{-\beta/2}u(1, t^{-\beta/2}x)$.

Introduction and Known Results

- Recently, Barlow, M.T. and Černý, J. 2009 obtained $B(E(t))$ as the scaling limit of a random walk in a random environment.
- More generally, for a uniformly elliptic operator L on a bounded domain $D \subset \mathbb{R}$, under suitable technical conditions and assuming Dirichlet boundary conditions, the diffusion equation $\partial u/\partial t = Lu$ governs a Markov process $Y(t)$ killed at the boundary, and the corresponding fractional diffusion equation $\partial^\beta u/\partial t^\beta = Lu$ governs the time-changed process $Y(E(t))$ (Meerschaert, M. M., Nane, E. and Vellaisamy, P. 2009).

Introduction and Known Results

- In some applications, waiting times between particle jumps evolve according to a more complicated process, which cannot be adequately described by a single power law.
- A mixture of power laws leads to a distributed-order fractional derivative in time (Chechkin, A. V., Gorenfo, R., and Sokolov, I. M. 2002, Mainardi, F., Mura, A., Pagnini, G. and Gorenflo, R. 2008, Mainardi, F., Mura, A., Gorenflo, R. and Stojanovic, M. 2007, Naber, M. 2004).
- An important application of distributed-order diffusions is to model ultraslow diffusion where a plume of particles spreads at a logarithmic rate (Meerschaert, M. M. and Scheffler, H. P. 2006, Y.G. Sinai 1982).

Introduction and Known Results

When L is the generator of a uniformly bounded and continuous semigroup on a Banach spaces, Baeumer and Meerschaert (2001) showed that the solution of

$$\partial^\beta u/\partial t^\beta = Lu$$

is analytic in a sectorial region. We extend the results of Baeumer and Meerschaert (2001) to distributed order fractional diffusion case.

Semigroups

- Let $||f||_1 = \int |f(x)|dx$ be the usual norm on the Banach space $L^1(\mathbb{R}^d)$ of absolutely integrable functions $f : \mathbb{R}^d \to \mathbb{R}$.
- A family of linear operators $\{T(t) : t \geq 0\}$ on a Banach space X such that $T(0)$ is the identity operator and $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$ is called a continuous convolution semigroup.
- If $||T(t)f|| \leq M||f||$ for all $f \in X$ and all $t \geq 0$ then the semigroup is uniformly bounded.
- If $T(t_n)f \to T(t)f$ in X for all $f \in X$ whenever $t_n \to t$ then the semigroup is strongly continuous.

Generator of Semigroups

- For any strongly continuous semigroup $\{T(t); t > 0\}$ on a Banach space X we define the generator

$$Lf = \lim_{t \to 0^+} \frac{T(t)f - f}{t} \text{ in } X \tag{1}$$

meaning that $||t^{-1}(T(t)f - f) - Lf|| \to 0$ in the Banach space norm.

- The domain $D(L)$ of this linear operator is the set of all $f \in X$ for which the limit in (1) exists.
- The domain $D(L)$ is dense in X, and L is closed, meaning that if $f_n \to f$ and $Lf_n \to g$ in X then $f \in D(L)$ and $Lf = g$ (see, for example Corollary I.2.5 in A. Pazy 1983).

Semigroups

Another consequence of $T(t)$ being a strongly continuous semigroup is that $u(t) = T(t)f$ solves the Cauchy problem

$$\frac{d}{dt}u(t) = Lu(t); \quad u(0) = f$$

for $f \in D(L)$. Furthermore, the integrated equation

$$T(t)f = L\int_0^t T(s)fds + f$$

holds for all $f \in X$ (see, for example, Theorem I.2.4 in A. Pazy 1983).

Distributed order fractional derivatives

The Caputo fractional derivative (Caputo, M., 1967) is defined for $0 < \beta < 1$ as

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(r,x)}{\partial r} \frac{dr}{(t-r)^\beta}. \tag{2}$$

Its Laplace transform

$$\int_0^\infty e^{-st} \frac{\partial^\beta u(t,x)}{\partial t^\beta} ds = s^\beta \tilde{u}(s,x) - s^{\beta-1} u(0,x) \tag{3}$$

where $\tilde{u}(s,x) = \int_0^\infty e^{-st} u(t,x) dt$ and incorporates the initial value in the same way as the first derivative.

Distributed order fractional derivatives

- The distributed order fractional derivative is

$$\mathbb{D}^{(\mu)}u(t,x) := \int_0^1 \frac{\partial^\beta u(t,x)}{\partial t^\beta}\mu(d\beta), \tag{4}$$

  where μ is a finite Borel measure with $\mu(0,1) > 0$.

- For a function $u(t,x)$ continuous in $t \geq 0$, the Riemann-Liouville fractional derivative of order $0 < \beta < 1$ is defined by

$$\left(\frac{\partial}{\partial t}\right)^\beta u(t,x) = \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t \frac{u(r,x)}{(t-r)^\beta}dr. \tag{5}$$

  Its Laplace transform

$$\int_0^\infty e^{-st}\left(\frac{\partial}{\partial t}\right)^\beta u(t,x)\,ds = s^\beta \tilde{u}(s,x). \tag{6}$$

Distributed order fractional derivatives

- If $u(\cdot, x)$ is absolutely continuous on bounded intervals (e.g., if the derivative exists everywhere and is integrable) then the Riemann-Liouville and Caputo derivatives are related by

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \left(\frac{\partial}{\partial t}\right)^\beta u(t,x) - \frac{t^{-\beta} u(0,x)}{\Gamma(1-\beta)}. \tag{7}$$

- The (extended) distributed order derivative is

$$\mathbb{D}_1^{(\mu)} u(t,x) := \int_0^1 \left[\left(\frac{\partial}{\partial t}\right)^\beta u(t,x) - \frac{t^{-\beta} u(0,x)}{\Gamma(1-\beta)}\right] \mu(d\beta), \tag{8}$$

which exists for $u(\cdot, x)$ continuous, and agrees with the usual definition (4) when $u(\cdot, x)$ is absolutely continuous.

Waiting time process

- For each $c > 0$, take a sequence of i.i.d. waiting times (J_n^c) and i.i.d. jumps (Y_n^c). Let $X^c(n) = Y_1^c + \cdots + Y_n^c$ be the particle location after n jumps, and $T^c(n) = J_1^c + \cdots + J_n^c$ the time of the nth jump.
- Suppose that $X^c([ct]) \Rightarrow A(t)$ and $T^c([ct]) \Rightarrow D_\psi(t)$ as $c \to \infty$, where the limits $A(t)$ and $D_\psi(t)$ are independent Lévy processes. Then [Meerschaert, M. M. and Scheffler, H. P., 2008, Theorem 2.1] shows that $X^c(N_t^c) \Rightarrow A(E_\psi(t))$, where

$$N_t^c = \max\{n \geq 0 : T^c(n) \leq t\}$$

is the number of jumps by time $t \geq 0$, and

$$E_\psi(t) = \inf\{\tau : D_\psi(\tau) > t\}. \tag{9}$$

Waiting time process

A specific mixture model from Meerschaert, M. M. and Scheffler, H. P. 2006 gives rise to distributed order fractional derivatives:

- Set (B_i), $0 < B_i < 1$, be i.i.d. random variables such that

$$P\{J_i^c > u | B_i = \beta\} = c^{-1} u^{-\beta},$$

  for $u \geq c^{-1/\beta}$.
- Then $T^c([ct]) \Rightarrow D_\psi(t)$, a subordinator with $\mathbb{E}[e^{-sD_\psi(t)}] = e^{-t\psi(s)}$, where

$$\psi(s) = \int_0^\infty (e^{-sx} - 1)\phi(dx). \tag{10}$$

- The associated Lévy measure is

$$\phi(t, \infty) = \int_0^1 t^{-\beta} \nu(d\beta), \tag{11}$$

  where ν is the distribution of B_i.

Lebesgue density of inverse subordinator

- Then, Theorem 3.10 in (Meerschaert, M. M. and Scheffler, H. P., 2006) shows that $c^{-1}N_t^c \Rightarrow E_\psi(t)$, where $E_\psi(t)$ is given by (9).
- The Lévy process $A(t)$ defines a strongly continuous convolution semigroup with generator L, and $A(E_\psi(t))$ is the stochastic solution to the distributed order-fractional diffusion equation

$$\mathbb{D}^{(\mu)}u(t,x) = Lu(t,x), \tag{12}$$

  where $\mathbb{D}^{(\mu)}$ is given by (4) with $\mu(d\beta) = \Gamma(1-\beta)\nu(d\beta)$.
- Theorem 3.1 in Meerschaert, M. M. and Scheffler, H. P., 2006 implies that $E_\psi(t)$ has a Lebesgue density

$$f_{E_\psi(t)}(x) = \int_0^t \phi(t-y,\infty)P_{D_\psi(x)}(dy). \tag{13}$$

Main Results

- Let $D_\psi(t)$ be a strictly increasing Lévy process (subordinator) with $\mathbb{E}[e^{-sD_\psi(t)}] = e^{-t\psi(s)}$, where the Laplace exponent $\psi(s)$ is given by (10).
- Let T be a uniformly bounded, strongly continuous semigroup on a Banach space. Let

$$S(t)f = \int_0^\infty (T(l)f) f_{E_\psi(t)}(l)dl \tag{14}$$

where $f_{E_\psi(t)}(l)$ is a Lebesgue density of $E_\psi(t)$ and

$$\int_0^\infty e^{-st} f_{E_\psi(t)}(l)dt = \frac{1}{s}\psi(s)e^{-l\psi(s)}.$$

Using Fubini's Theorem, we get

$$\int_0^\infty \psi(s)e^{-l\psi(s)} T(l)fdl = s\int_0^\infty e^{-st}S(t)fdt. \tag{15}$$

Main Results

- A *sectorial region* of the complex plane is defined as

$$\mathbb{C}(\alpha) := \{re^{i\theta} \in \mathbb{C} : \ r > 0, |\theta| < \alpha\}.$$

- A family of linear operators on a Banach space X is *strongly analytic in a sectorial region* if for some $\alpha > 0$ the mapping $t \to T(t)f$ has an analytic extension to the sectorial region $\mathbb{C}(\alpha)$ for all $f \in X$ (see, for example, section 3.12 in Hille, E. and Phillips, R. S., 1957).

Main Results

Theorem (Analytic Representation Theorem 2.6.1 Arendt, W., Batty, C., Hieber, M. and Neubrander, F., 2001)

Let $0 < \alpha \leq \frac{\pi}{2}, \omega \in \mathbb{R}$ *and* $q : (\omega, \infty) \to X$. *The following are equivalent:*

i) *There exists a holomorphic function* $f : \mathbb{C}(\alpha) \to X$ *such that* $sup_{z \in \mathbb{C}(\beta)} ||e^{-\omega z} f(z)|| < \infty$ *for all* $0 < \beta < \alpha$ *and* $q(\lambda) = \tilde{f}(\lambda)$ *for all* $\lambda > \omega$

ii) *The function* q *has a holomorphic extension* $\bar{q} : \omega + C(\alpha + \pi/2) \to X$ *such that* $sup_{\lambda \in \omega + \mathbb{C}(\gamma + \pi/2)} ||(\lambda - \omega)\bar{q}(\lambda)|| < \infty$ *for all* $0 < \gamma < \alpha$

Main Results

Let $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$. In the next theorem we consider the case where

$$\psi(s) = c_1 s^{\beta_1} + c_2 s^{\beta_2} + \cdots + c_n s^{\beta_n}.$$

In this case the Lévy subordinator can be written as

$$D_\psi(t) = (c_1)^{1/\beta_1} D^1(t) + (c_2)^{1/\beta_2} D^2(t) + \cdots + (c_n)^{1/\beta_n} D^n(t)$$

where $D^1(t), D^2(t), \cdots, D^n(t)$ are independent stable subordinators of index $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$.

Main Results

Theorem (Result 1)

Let $(X, ||.||)$ *be a Banach space and* L *be the generator of a uniformly bounded, strongly continuous semigroup* $\{T(t) : t \geq 0\}$.

- *Then the family* $\{S(t) : t \geq 0\}$ *of linear operators from* X *into* X *given by* (14) *is uniformly bounded and strongly analytic in a sectorial region.*
- *Furthermore,* $\{S(t) : t \geq 0\}$ *is strongly continuous and* $g(x, t) = S(t)f(x)$ *is a solution of*

$$\sum_{i=1}^{n} c_i \frac{\partial^{\beta_i} g(x,t)}{\partial t^{\beta_i}} = Lg(x,t); \; g(x,0) = f(x).$$

for $\beta_1 < \beta_2 < \cdots < \beta_n \in (0, 1)$.

Main Results

Proof.

- Let $q(s) = \int_0^\infty e^{-st} T(t) f dt$ and $F(s) = \int_0^\infty e^{-st} S(t) f dt$ for any $s > 0$, so that we can write (15) in the form

$$\psi(s) q(\psi(s)) = sF(s) \tag{16}$$

  for any $s > 0$.
- Since $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup with generator L, Theorem 1.2.4 (b) in A. Pazy, 1983) implies that $\int_0^t T(s) f ds$ is in the domain of the operator L and

$$T(t) f = L \int_0^t T(s) f ds + f.$$

Main Results

The next results provides an extension with subordinator $D_\mu(t)$ as the weighted average of an arbitrary number of independent stable subordinators. Let $E_\mu(t)$ be the inverse of the subordinator $D_\mu(t)$ with Laplace exponent

$$\psi(s) = \int_0^1 s^\beta d\mu(\beta)$$

where $supp\,\mu \subset (0,1)$.

Main Results

Theorem (Result 2)

Let $(X, ||\cdot||)$ be a Banach space and μ be a positive finite measure with supp $\mu \subset (0,1)$.

- *Then the family $\{S(t): t \geq 0\}$ of linear operators from X into X given by $S(t)f = \int_0^\infty (T(l)f) f_{E_\mu(t)}(l)dl$, is uniformly bounded and strongly analytic in a sectorial region.*
- *Furthermore, $\{S(t): t \geq 0\}$ is strongly continuous and $g(x,t) = S(t)f(x)$ is a solution of*

$$\int_0^1 \partial_t^\beta g(x,t)\mu(d\beta) = Lg(x,t);\ g(x,0) = f(x). \tag{17}$$

Open Problems and future works

- It looks like a challenging problem to extend the methods applied in the main results to more general time operators defined as $\psi(\partial_t) - \phi(t, \infty)$ where ψ is given by

$$\psi(s) = bs + \int_0^\infty (e^{-sx} - 1)\phi(dx), \qquad b \geq 0$$

- Meerschaert and Scheffler, 2008 define this operator by its Laplace transform as

$$\int_0^\infty e^{-st}\psi(\partial_t)g(t)dt = \psi(s)\tilde{g}(s).$$

- Likewise it looks like a challenging problem to extend the results in this paper to the case where $\psi(s) = (s + \lambda)^\beta - \lambda^\beta$ for $\lambda > 0$.
- This ψ gives rise to the so called tempered fractional derivative operator studied by (B. Baeumer and M. M. Meerschaert, 2010, Meerschaert, M. M., Nane, E. and Vellaisamy, P. 2011).

THANK YOU ALL FOR YOUR ATTENTION

QUESTIONS?