Second order chaos and processes on Heisenberg-like groups

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Smooth measures in finite dimensions

Definition A measure μ on $\mathbb{R}^n$ is said to be smooth if μ is absolutely continuous with respect to Lebesgue measure and the Radon-Nikodym derivative is smooth – that is,

$$\mu = \rho \, dm, \text{ for some } \rho \in C^\infty(\mathbb{R}^n, (0, \infty)).$$

Three examples on $\mathbb{R}^3$

1. Gaussian measure: In particular, for $t > 0$ and $\mu_t \sim \mathrm{Normal}(0, t)$, we have

$$d\mu_t(x) = \frac{1}{(2\pi t)^{3/2}} e^{-|x|^2/2t}\, dx.$$

Of course, $\mu_t = \mathrm{Law}(B_t)$ where $B_t = (B_t^1, B_t^2, B_t^3)$ is Brownian motion with generator $L = \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$.

Three examples on $\mathbb{R}^3$

2. elliptic "Heisenberg" example: Now let

$$\tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right)$$
$$\tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right)$$
$$\tilde{X}_3(x) = (0, 0, 1)$$

Note that for all $x \in \mathbb{R}^3$

$$\operatorname{span}\{\tilde{X}_1(x), \tilde{X}_2(x), \tilde{X}_3(x)\} = \mathbb{R}^3.$$

Consider the SDE

$$d\xi_t = \tilde{X}_1(\xi_t) \circ dB_t^1 + \tilde{X}_2(\xi_t) \circ dB_t^2 + \tilde{X}_3(\xi_t) \circ dB_t^3$$

with $\xi_0 = 0$.

Three examples on $\mathbb{R}^3$

2. elliptic "Heisenberg" example: Now let

$$\tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right) = \partial_1 - \frac{1}{2}x_2\partial_3$$
$$\tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right) = \partial_2 + \frac{1}{2}x_1\partial_3$$
$$\tilde{X}_3(x) = (0, 0, 1) = \partial_3$$

Then the solution to the SDE

$$d\xi_t = \tilde{X}_1(\xi_t) \circ dB_t^1 + \tilde{X}_2(\xi_t) \circ dB_t^2 + \tilde{X}_3(\xi_t) \circ dB_t^3,$$

is generated by the elliptic operator $L = \tilde{X}_1^2 + \tilde{X}_2^2 + \tilde{X}_3^2$, and $\mu_t = \mathrm{Law}(\xi_t)$ is a smooth measure on $\mathbb{R}^3$. We have

$$\xi_t = \left(B_t^1, B_t^2, B_t^3 + \frac{1}{2}\int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1\right).$$

Three examples on $\mathbb{R}^3$

3. hypoelliptic "Heisenberg" example: Again let

$$\tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right) = \partial_1 - \frac{1}{2}x_2\partial_3$$
$$\tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right) = \partial_2 + \frac{1}{2}x_1\partial_3$$
$$\tilde{X}_3(x) = (0, 0, 1) = \partial_3$$

Note that $[\tilde{X}_1, \tilde{X}_2] := \tilde{X}_1\tilde{X}_2 - \tilde{X}_2\tilde{X}_1 = \tilde{X}_3$. Thus, we could write

$$\mathrm{span}\{\tilde{X}_1(x), \tilde{X}_2(x), [\tilde{X}_1, \tilde{X}_2](x)\} = \mathbb{R}^3.$$

Consider the SDE

$$d\eta_t = \tilde{X}_1(\eta_t) \circ dB_t^1 + \tilde{X}_2(\eta_t) \circ dB_t^2$$

with $\eta_0 = 0$.

Three examples on $\mathbb{R}^3$

3. hypoelliptic "Heisenberg" example: Again let

$$\tilde{X}_1(x) = \left(1, 0, -\frac{1}{2}x_2\right) = \partial_1 - \frac{1}{2}x_2\partial_3$$
$$\tilde{X}_2(x) = \left(0, 1, \frac{1}{2}x_1\right) = \partial_2 + \frac{1}{2}x_1\partial_3$$
$$\tilde{X}_3(x) = (0, 0, 1) = \partial_3$$

Thus, Hörmander's theorem implies that $\mathcal{L} = \tilde{X}_1^2 + \tilde{X}_2^2$ is a "hypoelliptic" operator in the sense that the diffusion

$$d\eta_t = \tilde{X}_1(\eta_t) \circ dB_t^1 + \tilde{X}_2(\eta_t) \circ dB_t^2,$$

has a smooth measure $\nu_t = \mathrm{Law}(\eta_t)$ on $\mathbb{R}^3$. Again, we may solve this SDE explicitly as

$$\eta_t = \left(B_t^1, B_t^2, \frac{1}{2}\int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1\right).$$

An aside on Lévy Area

The process given by

$$A(t) := \int_0^t B_s^1\, dB_s^2 - B_s^2 dB_s^1$$

is called the stochastic Lévy area of (B^1, B^2). It is well known that

$$\{A(t)\} \stackrel{d}{=} \left\{ B\left(\int_0^t (B_s^1)^2 + (B_s^2)^2\, ds \right) \right\}$$

where B is a real BM independent of (B_1, B_2). (See, for example, Ikeda & Watanabe.)

LIL and FLIL have been proved for $\{A(t)\}$ and generalizations of this process via this representation and small ball estimates: see Shi (1994), Rémillard (1994), and Kuelbs & Li (2005).

A geometric interpretation: Heisenberg group

Let $\mathfrak{g} = \mathrm{span}\{X_1, X_2, X_3\}$ so that $[X_1, X_2] = X_3$. Then $G \cong \mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ with

$$x \cdot x' = \left(x_1 + x_1', x_2 + x_2', x_3 + x_3' + \frac{1}{2}(x_1 x_2' - x_2 x_1') \right).$$

Then $\tilde{X}_i$ is the unique left invariant vector field on G such that $\tilde{X}_i(e) = X_i$. We may think of

$$d\xi_t = \xi_t \circ dB_t := \sum_{i=1}^{3} \tilde{X}_i(\xi_t) \circ dB_t^i$$

as "rolling" the flat BM $B_t = B_t^1 X_1 + B_t^2 X_2 + B_t^3 X_3$ onto G. (Similarly for η with $B_t = B_t^1 X_1 + B_t^2 X_2$.)

We will call ξ_t and η_t Brownian motion and hypoelliptic Brownian motion, respectively, on G.

Smooth measures on Lie groups

Let G be a Lie group with identity e and Lie algebra $\mathfrak{g}$ with $\dim(\mathfrak{g}) = n$.

Suppose $\operatorname{span}(\{X_i\}_{i=1}^n) = \mathfrak{g}$ and let

$$L = \sum_{i=1}^{n} \tilde{X}_i^2$$

where $\tilde{X}$ is the unique left invariant v.f. such that $\tilde{X}(e) = X$. Then L is an elliptic operator and

$$d\xi_t = \xi_t \circ dB_t := \sum_{i=1}^{n} \tilde{X}_i(\xi_t) \circ dB_t^i$$

with $\xi_0 = e$, has a smooth law on G.

Smooth measures on Lie groups

More generally, suppose $\{X_i\}_{i=1}^k \subset \mathfrak{g}$ satisfies

$$\mathrm{span}\{X_i, [X_{i_1}, X_{i_2}], \ldots, [X_{i_1}, [\cdots, [X_{i_{r-1}}, X_{i_r}]]]\} = \mathfrak{g}. \qquad \text{(HC)}$$

Then $L = \sum_{i=1}^k \tilde{X}_i^2$ is hypoelliptic, and

$$d\xi_t = \xi_t \circ dB_t := \sum_{i=1}^k \tilde{X}_i(\xi_t) \circ dB_t^i$$

with $\xi_0 = e$, has a smooth law on G (where now B_t is BM on $\mathfrak{g}_0 := \mathrm{span}(\{X_i\}_{i=1}^k))$.

that is, $\exists\ 0 < p_t \in C^\infty(G)$ such that

$$d\nu_t := \mathrm{Law}(\xi_t) = p_t(\cdot)\, d(Haar).$$

We call p_t the heat kernel and ν_t heat kernel measure.

Smooth measures in infinite dimensions

Definition^1 A measure μ on $\mathbb{R}^n$ is said to be smooth if μ is absolutely continuous with respect to Lebesgue measure and the Radon-Nikodym derivative is smooth – that is,

$$\mu = \rho\, dm, \text{ for some } \rho \in C^\infty(\mathbb{R}^n, (0,\infty)).$$

Definition^2 A measure μ on $\mathbb{R}^n$ is said to be smooth if, for any multi-index α, there exists a function $g_\alpha \in C^\infty(\mathbb{R}^n) \cap L^{\infty-}(\mu)$ such that

$$\int_{\mathbb{R}^n} (-D)^\alpha f\, d\mu = \int_{\mathbb{R}^n} f g_\alpha\, d\mu, \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

Fact: Definition^1 $\iff$ Definition^2

A first step to smoothness: Quasi-invariance

Definition A measure μ on Ω is quasi-invariant under a transformation $T : \Omega \to \Omega$ if μ and $\mu \circ T^{-1}$ are mutually absolutely continuous.

In particular, we will be interested in quasi-invariance under transformations of the type

$$T = T_h = \text{translation by an element } h \in \Omega_0 \subset \Omega,$$

where Ω_0 is some distinguished subset of Ω.

Quasi-invariance: *The finite-dimensional examples*

Recall the standard Gaussian measure on $\mathbb{R}^n$ given by $d\mu(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}\, dx$. Then, for any $y \in \mathbb{R}^n$,

$$d\mu^y(x) := d\mu(x-y) = \frac{1}{(2\pi)^{n/2}} e^{-|x-y|^2/2}\, dx$$
$$= e^{-|y|^2/2+\langle x,y\rangle}\, d\mu(x)$$

More generally, for ν a smooth measure on a (fin-dim) Lie group G with density $p > 0$ (that is, $d\nu(x) = p(x)\, d(Haar)(x)$), for any $y \in G$,

$$d\nu^y(x) := d(\nu \circ R_y^{-1})(x)$$
$$= p(xy^{-1})\, d(Haar)(x) = \frac{p(xy^{-1})}{p(x)}\, d\nu(x).$$

Quasi-invariance: *Cameron-Martin theorem*

Let

$$W(\mathbb{R}^n) = \{w : [0,T] \to \mathbb{R}^n : w \text{ is continuous and } w(0) = 0\}$$

equipped with Wiener measure μ and

$$H(\mathbb{R}^n) = \{h \in W(\mathbb{R}^n) : h \text{ is abs cts and } \int_0^T |\dot{h}(t)|^2 \, dt < \infty\}.$$

Then, for any $y \in H(\mathbb{R}^n)$,

$$d\mu^y(x) := d\mu(x-y) = e^{-|y|_H^2/2 + \text{``}\langle x,y\rangle\text{''}} \, d\mu(x).$$

Moreover, if $y \notin H$, then $\mu_y \perp \mu$.

More generally, this holds for any abstract Wiener space (W, H, μ).

∞-dimensional curved example

Theorem (*Shigekawa, 1982*)
Let G be a (fin dim) compact group with Lie algebra $\mathfrak{g}$. Let $W(G)$ be path space on G equipped with "Wiener measure" μ, and let $H(G)$ denote the space of finite-energy paths on G. Then μ is quasi-invariant under translation by elements of $H(G)$.

More generally, a Cameron-Martin type quasi-invariance theorem holds for paths on a Riemannian manifold.

See Driver (1992), Hsu (1995,2002), Enchev & Stroock (1995), and Hsu & Ouyang (2010), as well as Airault & Malliavin (2006), Driver & Gordina (2008), et al for other infinite-dimensional elliptic examples.

∞-dimensional Heisenberg-like groups

Definition Let (W, H, μ) be an abstract Wiener space and $\mathbf{C}$ be a finite-dimensional inner product space.

Then $\mathfrak{g} = W \times \mathbf{C}$ is a Heisenberg-like Lie algebra if

1. $[W, W] \subset \mathbf{C}$ and $[W, \mathbf{C}] = [\mathbf{C}, \mathbf{C}] = 0$, and
2. $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}$ is continuous.

Let G denote $W \times \mathbf{C}$ when thought of as the Lie group with multiplication given by

$$gg' = g + g' + \frac{1}{2}[g, g'].$$

Let $\mathfrak{g}_{CM}$ denote $H \times \mathbf{C}$ when thought of as a Lie subalgebra of $\mathfrak{g}$, and let G_{CM} denote $H \times \mathbf{C}$ when thought of as a subgroup of G.

An elliptic q.i. theorem on Heisenberg-like groups

Let G be an infinite-dimensional Heisenberg-like group.

Definition Let $b_t = (B_t^W, B_t^{\mathbf{C}})$ be BM on $\mathfrak{g}$. Then

$$d\xi_t = \xi_t \circ db_t, \qquad \text{with } \xi_0 = e,$$

is BM on G. This may be solved explicitly as

$$\xi_t = \left(B_t^W, B_t^{\mathbf{C}} + \frac{1}{2} \int_0^t [B_t^W, dB_t^W] \right).$$

For $t > 0$, let $\mu_t = \mathrm{Law}(\xi_t)$ denote the heat kernel measure on G.

An aside on second-order chaos

Consider the case where $\dim(\mathbf{C}) = 1$, and let $\{h_i\}_{i=1}^{\infty}$ be an ONB of H in W^*. Consider the process

$$
\begin{aligned}
A(t) &:= \int_0^t [B_t^W, dB_t^W] \\
\text{``} = \text{''} & \int_0^t \left[\sum_{i=1}^{\infty} B_s^i h_i, \sum_{j=1}^{\infty} dB_s^j h_j \right] \\
&= \sum_{i<j} [h_i, h_j] \int_0^t B_s^i dB_s^j - B_s^j dB_s^i \overset{d?}{=} B(C(t))
\end{aligned}
$$

where B is an independent real BM and

$$
C(t) = \sum_{i=1}^{\infty} \alpha_i^2 \int_0^t (B_s^{2i-1})^2 + (B_s^{2i})^2 \, ds.
$$

And if so, can one use the arguments from Kuelbs & Li (2005) to prove LIL, FLIL?

An elliptic q.i. theorem on Heisenberg-like groups

Let G be an infinite-dimensional Heisenberg-like group.

Definition Let $b_t = (B_t^W, B_t^{\mathbf{C}})$ be BM on $\mathfrak{g}$. Then

$$d\xi_t = \xi_t \circ db_t, \qquad \text{with } \xi_0 = e,$$

is BM on G. For $t > 0$, let $\mu_t = \mathrm{Law}(\xi_t)$ denote the heat kernel measure on G.

Theorem (*Driver and Gordina, 2008*)
For all $y \in G_{CM}$ and $t > 0$, μ_t is quasi-invariant under left and right translations by y. Moreover, for all $q \in (1, \infty)$,

$$\left\| \frac{d(\mu_t \circ R_y^{-1})}{d\mu_t} \right\|_{L^q(G,\nu_t)} \le \exp\left(\frac{k(q-1)}{2(e^{kt}-1)} d(e,y)^2 \right)$$

and similarly for $\frac{d(\mu_t \circ L_y^{-1})}{d\mu_t}$, where "$\mathrm{Ric} \ge k$".

Sketch of elliptic proof:

Define a class of finite-dimensional "projection" groups G_P, so that for

$$d\xi_t^P = \xi_t^P \circ dPb_t$$

we have $\xi_t^{P_n} \to \xi_t$, and prove that "$\sup_P \mathrm{Ric}^P \geq k$". Then

$$\text{“}\mathrm{Ric}^P \geq k\text{”} \implies \text{log Sobolev inequality}$$

$$\implies \text{Wang/Integrated Harnack inequality:}$$

For all $y \in G_P$ and $q \in (1,\infty)$,

$$\left(\int_G \left[\frac{p_t^P(xy^{-1})}{p_t^P(x)}\right]^q p_t^P(x)\,dx\right)^{1/q} \leq \exp\left(\frac{k(q-1)}{2(e^{kt}-1)} d^P(e,y)^2\right).$$

Recall that, for $d\mu_t^P(x) = p_t^P(x)\,dx$,

$$\frac{d\mu_t^P \circ R_y^{-1}}{d\mu_t^P} = \frac{p_t^P(xy^{-1})}{p_t^P(x)} =: J_t^P(x,y).$$

Sketch of elliptic proof:

So fix a projection P_0 and let $\{P_n\}_{n=0}^{\infty}$ so that $P_n \uparrow I|_{G_{CM}}$. Let $y \in G_0 \subset G_{P_n}$. Then, for any $f \in BC(G)$ and $q' \in (1,\infty)$,

$$\int_{G_n} |(f \circ i_n)(xy)|\, d\mu_t^{P_n}(x) = \int_{G_n} J_t^{P_n}(x,y)|(f \circ i_n)(x)|\, d\mu_t^{P_n}(x)$$
$$\leq \|f \circ i_n\|_{L^{q'}(G_n, \mu_t^{P_n})} \exp\left(\frac{k(q-1)}{2(e^{kt}-1)} d^{P_n}(e,y)^2 \right).$$

Taking the limit as $n \to \infty$,

$$\int_G f(xy)\, d\mu_t(x) \leq \|f\|_{L^{q'}(G,\mu_t)} \exp\left(\frac{k(q-1)}{2(e^{kt}-1)} d(e,y)^2 \right).$$

Thus, $J_t(\cdot, y) := d(\mu_t \circ R_y^{-1})/d\mu_t$ exists in L^q for all $q \in (1,\infty)$. ∎

Problems with $\mathrm{Ric} \geq k$ in hypoelliptic setting

Let

$$\Gamma(f,g) = \frac{1}{2}\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f$$

$$\Gamma_2(f) = \frac{1}{2}\mathcal{L}\Gamma(f,f) - \Gamma(f,\mathcal{L}f).$$

Problems with $\mathrm{Ric} \geq k$ in hypoelliptic setting

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$$\Gamma_2(f) = \frac{1}{2}\mathcal{L}\Gamma(f,f) - \Gamma(f,\mathcal{L}f).$$

In particular, if $\mathcal{L} = \sum_{i=1}^{k} \tilde{X}_i^2$, then

$$\Gamma(f,g) = \nabla f \cdot \nabla g = \sum_{i=1}^{k} (\tilde{X}_i f)(\tilde{X}_i g).$$

Problems with $\mathrm{Ric} \geq k$ in hypoelliptic setting

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In particular, if $\mathcal{L} = \sum_{i=1}^{k} \tilde{X}_i^2$, then

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We have the following **FACT:** "$\mathrm{Ric} \geq k$" $\iff$

$$\Gamma_2(f) \geq k\Gamma(f,f), \quad \forall f \in C^\infty(G). \qquad \text{(CDI)}$$

Problems with $\mathrm{Ric} \geq k$ in hypoelliptic setting

Consider the Heisenberg group case again.

$$\tilde{X}(x,y,z) = \partial_x - \frac{1}{2} y \partial_z \qquad \tilde{Y}(x,y,z) = \partial_y + \frac{1}{2} x \partial_z$$

Then $\mathcal{L} = \tilde{X}^2 + \tilde{Y}^2$ and

$$\Gamma(f) := \Gamma(f,f) = (\tilde{X} f)^2 + (\tilde{Y} f)^2$$

$$\Gamma_2(f) = \frac{1}{2} \mathcal{L} \Gamma(f) - \Gamma(f, \mathcal{L} f)$$

Note that

$$\Gamma(f)(0) = f_x^2 + f_y^2$$

$$\Gamma_2(f)(0) = \sum_{i,j=1}^{2} |\partial_i \partial_j f(0)|^2 + \frac{1}{2} f_z^2(0) + 2(f_y f_{x,z} - f_x f_{y,z})(0).$$

Then there is **no** constant $k \in \mathbb{R}$ so that

$$\Gamma_2(f)(0) \geq k \Gamma(f)(0), \quad \forall f \in C^\infty(G).$$

A replacement for $\mathrm{Ric} \geq k \iff \Gamma_2 \geq k\Gamma$?

Suppose G is a (fin-dim) Lie group with $\mathrm{Lie}(\{X_i\}_{i=1}^k) = \mathfrak{g}$, and let $\{Z_i\}_{i=1}^d$ be an ONB of $\mathrm{span}(\{X_i\}_{i=1}^k)^\perp$. Define

$$\Gamma^Z(f,g) := \sum_{i=1}^d (\tilde{Z}_i f)(\tilde{Z}_i g)$$

$$\Gamma_2^Z(f) := \frac{1}{2}\mathcal{L}\Gamma^Z(f) - \Gamma^Z(f, \mathcal{L}f).$$

Suppose there exists $\alpha, \beta > 0$ such that, for all $\lambda > 0$,

$$\Gamma_2(f) + \lambda\Gamma_2^Z(f) \geq \alpha\Gamma^Z(f) - \frac{\beta}{\lambda}\Gamma(f). \tag{GCDI}$$

A replacement for $\mathrm{Ric} \geq k \iff \Gamma_2 \geq k\Gamma$?

Suppose G is a (fin-dim) Lie group with $\mathrm{Lie}(\{X_i\}_{i=1}^k) = \mathfrak{g}$, and let $\{Z_i\}_{i=1}^d$ be an ONB of $\mathrm{span}(\{X_i\}_{i=1}^k)^\perp$. Define

$$\Gamma^Z(f,g) := \sum_{i=1}^d (\tilde{Z}_i f)(\tilde{Z}_i g)$$

$$\Gamma_2^Z(f) := \frac{1}{2}\mathcal{L}\Gamma^Z(f) - \Gamma^Z(f,\mathcal{L}f).$$

Suppose there exists $\alpha, \beta > 0$ such that, for all $\lambda > 0$,

$$\Gamma_2(f) + \lambda\Gamma_2^Z(f) \geq \alpha\Gamma^Z(f) - \frac{\beta}{\lambda}\Gamma(f). \tag{GCDI}$$

Then (GCDI) $\Longrightarrow$ reverse log Sobolev inequality

$\Longrightarrow$ Wang/Integrated Harnack inequality:

For all $y \in G$ and $q \in (1,\infty)$,

$$\left(\int_G \left[\frac{p_t(xy^{-1})}{p_t(x)}\right]^q p_t(x)\,dx\right)^{1/q} \leq \exp\left(\left(1+\frac{8\beta}{\alpha}\right)\frac{1+q}{4t}d_h(e,y)^2\right).$$

(GCDI) for 3-dim Heisenberg group

$$
\begin{aligned}
\Gamma(f) &= (Xf)^2 + (Yf)^2 \\
\Gamma^Z(f) &= (Zf)^2 \\
\Gamma_2(f) &= (X^2f)^2 + (Y^2f)^2 + \frac{1}{2}((XY+YX)f)^2 \\
&\qquad + \frac{1}{2}(Zf)^2 - 2(Xf)(YZf) + 2(Yf)(XZf) \\
\Gamma_2^Z(f) &= \frac{1}{2}\mathcal{L}\Gamma^Z(f) - \Gamma^Z(f, \mathcal{L}f) \\
&= \frac{1}{2}(X^2+Y^2)(Zf)^2 - (Zf)\cdot Z(X^2f + Y^2f) \\
&= (XZf)^2 + (YZf)^2 + (Zf)(ZX^2f + ZY^2f) \\
&\qquad - (Zf)(ZX^2f + ZY^2f) \\
&= (XZf)^2 + (YZf)^2
\end{aligned}
$$

(GCDI) for 3-dim Heisenberg group

Note that

$$\Gamma_2(f) \geq \frac{1}{2}(Zf)^2 - 2(Xf)(YZf) + 2(Yf)(XZf)$$

For example, taking $\lambda = 1$

$$\begin{aligned}
&\Gamma_2(f) + \Gamma_2^Z(f) \\
&\quad \geq \frac{1}{2}(Zf)^2 - 2(Xf)(YZf) + 2(Yf)(XZf) + (XZf)^2 + (YZf)^2 \\
&\quad = \frac{1}{2}(Zf)^2 + (Xf - YZf)^2 - (Xf)^2 + (Yf + XZf)^2 - (Yf)^2 \\
&\quad \geq \frac{1}{2}(Zf)^2 - (Xf)^2 - (Yf)^2 \\
&\quad = \frac{1}{2}\Gamma^Z(f) - \Gamma(f)
\end{aligned}$$

So (GCDI) holds with $\alpha = \frac{1}{2}$ and $\beta = 1$.

Hypoelliptic BM on Heisenberg-like groups

Let $G = W \times \mathbf{C}$ be an infinite-dimensional Heisenberg-like Lie algebra such that

$$[W, W] = \mathbf{C}.$$

Then, for B_t^W BM on W, the "hypoelliptic" Brownian motion on G is the solution to

$$d\eta_t = \eta_t \circ dB_t^W, \qquad \text{with } \eta_0 = e.$$

This may be solved explicitly as

$$\eta_t = B_t^W + \frac{1}{2}\int_0^t [B_s^W, dB_s^W] = \left(B_t^W, \frac{1}{2}\int_0^t [B_s^W, dB_s^W] \right).$$

Let $\nu_t = \mathrm{Law}(\eta_t)$ be the "hypoelliptic" heat kernel measure.

Theorem (*Baudoin, Gordina, M. 2011*)

For all $y \in G_{CM}$ and $t > 0$, ν_t is quasi-invariant under left and right translations by y. Moreover, for all $q \in (1, \infty)$,

$$\left\| \frac{d(\nu_t \circ R_y^{-1})}{d\nu_t} \right\|_{L^q(G,\nu_t)} \leq \exp\left(\left(1 + \frac{8\|[\cdot,\cdot]\|^2}{\rho_2} \right) \frac{1+q}{4t} d_h(e,y)^2 \right)$$

and similarly for $\frac{d(\nu_t \circ L_y^{-1})}{d\nu_t}$, where

$$\|[\cdot,\cdot]\|^2 := \|[\cdot,\cdot]\|^2_{H^* \otimes H^* \otimes \mathbf{C}} := \sum_{i,j=1}^{\infty} \sum_{\ell=1}^{d} \langle [e_i, e_j], f_\ell \rangle^2_{\mathbf{C}},$$

$$\rho_2 := \inf \left\{ \sum_{i,j=1}^{\infty} \left(\sum_{\ell=1}^{d} \langle [e_i, e_j], f_\ell \rangle_{\mathbf{C}} x_\ell \right)^2 : \sum_{\ell=1}^{d} x_\ell^2 = 1 \right\},$$

and d_h is the horizontal distance on G_{CM}.

Sketch of proof:

We prove that for all $\lambda > 0$

$$\Gamma_{2,P}(f) + \lambda \Gamma^Z_{2,P}(f) \geq \rho_{2,P} \Gamma^Z_P(f) - \frac{\|[\cdot,\cdot]\|_P^2}{\lambda} \Gamma_P(f).$$

Thus, we have that for all projections P

$$\left(\int_G \left[\frac{p_t^P(xy^{-1})}{p_t^P(x)} \right]^q p_t^P(x)\, dx \right)^{1/q} \leq \exp\left(\left(1 + \frac{8\|[\cdot,\cdot]\|_P^2}{\rho_{2,P}} \right) \frac{1+q}{4t} d_h^P(e,y)^2 \right).$$

Sketch of proof:

Using this integrated Harnack inequality in the same limiting argument as in the elliptic case completes the proof (modulo topological considerations):

Fix a projection P_0 and let $\{P_n\}_{n=0}^{\infty}$ so that $P_n \uparrow I|_{G_{CM}}$. Let $y \in G_0$. Then, for any $f \in BC(G)$ and $q' \in (1, \infty)$,

$$\int_{G_n} |(f \circ i_n)(xy)| \, d\nu_t^n(x) = \int_{G_n} J_t^n(x,y) |(f \circ i_n)(x)| \, d\nu_t^n(x)$$
$$\leq \|f \circ i_n\|_{L^{q'}(G_n, \nu_t^n)} \exp\left(\left(1 + \frac{8\|[\cdot,\cdot]\|_n^2}{\rho_{2,n}}\right) \frac{1+q}{4t} d_h^n(e,y)^2\right).$$

Taking the limit as $n \to \infty$,

$$\int_G f(xy) \, d\nu_t(x) \leq \|f\|_{L^{q'}(G,\nu_t)} \exp\left(\left(1 + \frac{8\|[\cdot,\cdot]\|^2}{\rho_2}\right) \frac{1+q}{4t} d_h(e,y)^2\right).$$

Thus, $J_t(\cdot, y) := d(\nu_t \circ R_y^{-1})/d\nu_t$ exists in L^q, $\forall\ q \in (1, \infty)$. ■