Semiparametric bounds on completely monotone functions

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Motivation

Recall: A function $f(s)$ is **completely monotone** function on $[0, \infty)$, if it has a property $(-1)^n f^{(n)}(s) \geq 0$ for all nonnegative integers n and all $s \geq 0$.

Bernstein's characterization states that a function $f(s)$ is completely monotone function on $[0, \infty)$ if and only if $f(s)$ is Laplace transform of a nonnegative measure, that is , it can be expressed as

$$f(s) = \int_0^\infty e^{-xs} d\sigma(x)$$

where σ is a nonnegative measure.

For the proof of Bernstein's characterization, see Feller (1971), Korenblum, etc.(1993), Schilling, Song and Vondraček (2010).

Completely monotone function has been a subject of intense research activity. Its remarkable applications can be seen in various fields, such as potential theory and physics, large and small deviation, numerical analysis, stieltjes transform, positive stable distribution, unimodal distribution, Bernstin function, Tauberian theorem, k-monotone density, special functions, etc. For interested reader, we refer to Kanter (1975), Alzer and Berg (2002), Gao and Wellner (2009), Gao, Li and Wellner (2010).

References

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Our Aim

In this talk, for given any random variable $S \geq 0$ and completely monotone function $f(s)$, various bounds are derived on the mean and variance of $f(S)$. The techniques are based on domination by exponential functions, Cauchy-Schwarz inequality and symmetrization method for variance.

Proposition 1 *Let r.v. $X \geq 0$ with c.d.f. $F_X(x)$. If c.d.f. of r.v. $S \geq 0$ is $F_S(s) = 1 - \int_0^\infty e^{-xs} dF_X(x)$, then for $\gamma > 0$, $\mathbb{E}\, S^\gamma = \Gamma(\gamma+1)\mathbb{E}\,(1/X^\gamma)$, where Γ is gamma function.*

- $\mathbb{E}\, S^\gamma = \int_0^\infty \gamma t^{\gamma-1}\mathbb{P}(S > t)dt.$

- for $x \geq 0$, $x^{-\gamma} = \Gamma(\gamma)^{-1} \int_0^\infty t^{\gamma-1} e^{-tx} dt.$

Proposition 2 *Let S is exponential r.v. with parameter λ and a function $f(s)$ is completely monotonic. If there exists $T > 0$ such that $f(s) \geq e^{-Ts}$ holds at one point $s = 1/(\lambda + T)$, then $\mathbb{E}\, f(S) \geq \max(f(1/\lambda), \lambda/(T+\lambda))$.*

Proposition 3 *Let S is exponential r.v. with parameter λ and a function $f(s)$ is completely monotonic. If there exists $T > 0$ such that $f(s) \leq e^{-Ts}$ hold at two points $s = 0$ and $s = b \geq 2/\lambda$, then*

$$\mathbb{E}\, f(S) \leq \min(f(0), \frac{\lambda}{\lambda + T}).$$

Proposition 4 *Let a function $f(s)$ is completely monotonic. If $f(0)$ is finite, then for any r.v. $S \geq 0$,*

$$\mathit{Var}(f(S)) \leq f(0) \cdot \int_0^\infty \mathit{Var}(e^{-xS}) d\sigma(x).$$

Proposition 5 *For any random variables $X \geq 0$ and $S \geq 0$, then*

$$\mathit{Var}\,\mathbb{E}\,(e^{-XS}|S) \leq \mathbb{E}\,\mathit{Var}(e^{-XS}|X).$$

Proposition 6 *Let a function $f(s)$ is completely monotonic. If S is Poisson r.v. with parameter λ, then $Var(f(S)) \leq \min(f^2(0) - f(0)f(2\lambda), f^2(0) - f^2(\lambda))$.*

Proposition 7 *Let S is exponential r.v. with parameter λ and a function $f(s)$ is completely monotonic. If there exists $T \geq (1+\sqrt{5})\lambda/2$ such that $f(s) \leq e^{-Ts}$ hold at two points $s = 0$ and $s = b \geq 3(3-\sqrt{5})/(2\lambda)$, then*

$$Var(f(S)) \leq \frac{\lambda T^2}{(2T+\lambda)(T+\lambda)^2}.$$

Proof.

- $\mathrm{Var}(e^{-xS}) = \frac{\lambda}{2x+\lambda} - (\frac{\lambda}{x+\lambda})^2$.

- finding "good" functions $g(x) = A + Be^{-bx}, x \geq 0$, such that $\mathrm{Var}(e^{-xS}) \leq g(x)$.

- Let $w(x) = 2\lambda x(x^2 - \lambda x - \lambda^2)(2x+\lambda)^{-2}(x+\lambda)^{-3}e^{bx}$, $B = w(T)/b$, $A = \mathrm{Var}(e^{-TS}) - w(T)/e^{-bT}/b$.

- $\mathrm{Var}(f(S)) \leq A + Be^{-Tb}$.

Thank you very much !