Ten Lectures on Small Value Probabilities and Applications

L9: Existence of Constants and Comparisons

Wenbo V. Li

University of Delaware

http://www.math.udel.edu/~wli
wli@math.udel.edu

CBMS Lectures at UAH, June 4-8, 2012

A scaling argument, similar to the well known sub-additive method, is considered for the fractional Brownian motion under the sup-norm and L_p-norm. Other comparison results are also discussed in order to preserve the constants in small value problems.

Scaling Lemma

Let X be a random element in a normed spece $(E, \|\cdot\|)$ and $F(x) = \mathbb{P}(\|X| \le x)$ for $x > 0$. Assuming $F(0) = 0$. If for any $x > 0$ and positive integer $n \ge 1$,

$$\log F(x) \le n \log F(n^{\alpha} x)$$

for some $\alpha > 0$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{1/\alpha} \log F(\varepsilon) = \inf_{\varepsilon > 0} \varepsilon^{1/\alpha} \log F(\varepsilon) < 0.$$

•Sub-additive lemma: If $a_{m+n} \le a_m + a_n$, for all $m, n \ge 1$, then

$$\lim_{n \to \infty} \frac{a_n}{n} \to \inf_{n \ge 1} \frac{a_n}{n}.$$

•Multiplicative form: $a_n = \log b_n$, $b_{m+n} \le b_m \cdot b_n$.
Ex: For i.i.d. sum S_n, we have

$$\lim_{n \to 0} \frac{1}{n} \mathbb{P}(S_n \le na) = -I(a), \quad a \le \mathbb{E}\, X_1.$$

Pf: Let

$$
\begin{aligned}
l &= \liminf_{\varepsilon\to 0} \varepsilon^{1/\alpha} \log F(\varepsilon) \\
L &= \limsup_{\varepsilon\to 0} \varepsilon^{1/\alpha} \log F(\varepsilon).
\end{aligned}
$$

Also let $\{a_n\}$, $\{b_n\}$ be two positive sequences such that $a_n \to 0$, $b_n \to 0$, $a_n b_n^{-1} \to \infty$ as $n \to \infty$, and

$$
\lim_{n\to\infty} a_n^{1/\alpha} \log F(a_n) = l, \quad \lim_{n\to\infty} b_n^{1/\alpha} \log F(b_n) = L.
$$

Then

$$
\begin{aligned}
\log F(b_n) &\le [(a_n b_n^{-1})^{1/\alpha}] \log F(\left[(a_n b_n^{-1})^{1/\alpha}\right]^{\alpha} b_n) \\
&\le [(a_n b_n^{-1})^{1/\alpha}] \log F(a_n)
\end{aligned}
$$

where $[x]$ denotes the greatest integer less than x. Hence

$$
b_n^{1/\alpha} \log F(b_n) \le (b_n a_n^{-1})^{1/\alpha} \cdot [(a_n b_n^{-1})^{1/\alpha}] \cdot a_n^{1/\alpha} \log F(a_n)
$$

implying $L \le l$ and consequently $L = l$.

Now for $\varepsilon > 0$ small and any fixed $x > 0$, there exists an integer $k \geq 1$ such that

$$x(k+1)^{-\alpha} \leq \varepsilon < xk^{-\alpha}.$$

Thus we have

$$\begin{aligned}
\varepsilon^{1/\alpha} \log F(\varepsilon) &\leq \varepsilon^{1/\alpha} \log F(xk^{-\alpha}) \\
&\leq \varepsilon^{1/\alpha} k \log F(x) \\
&\leq (k+1)^{-1} k \cdot x^{1/\alpha} \log F(x).
\end{aligned}$$

Hence it follows that

$$\lim_{\varepsilon \to 0} \varepsilon^{1/\alpha} \log F(\varepsilon) \leq x^{1/\alpha} \log F(x)$$

which clearly implies the statement.

A Refined Scaling Lemma

Let $\phi(x) > 0$ be a non-increasing function on $(0, \infty)$ with $\phi(0) = \infty$. If for any $x > 0$ small, $\delta > 0$ small and integer n large, it holds

$$\phi(x) \leq f_\delta(n)\phi(\lambda_\delta nx) + g_\delta(n)\psi_\delta(nx) \qquad (*)$$

or

$$\phi(x) \geq f_\delta(n)\phi(\lambda_\delta nx) - g_\delta(n)\psi_\delta(nx) \qquad (**)$$

with $f_\delta(n) \geq 0$, $g_\delta(n) \geq 0$, $\psi_\delta(x) \geq 0$ and

$$\lim_{\delta\to 0} \lambda_\delta = 1, \quad \lim_{\delta\to 0} \limsup_{n\to\infty} n^{-1} f_\delta(n) = 1$$

and

$$\lim_{\delta\to 0} \left(\limsup_{n\to\infty} n^{-1} g_\delta(n) \cdot \limsup_{x\to 0} x\psi_\delta(x) \right) = 0.$$

Then $\lim_{x\to 0} x\phi(x)$ exists and $0 \leq \lim_{x\to 0} x\phi(x) < \infty$ if (*) holds, $0 < \lim_{x\to 0} x\phi(x) \leq \infty$ if (**) holds. Furthermore, if in addition $\lim_{\delta\to 0} \limsup_{n\to\infty} n^{-1} g_\delta(n) = 0$, then $\lim_{x\to 0} x\phi(x) = \inf_{x>0} x\phi(x)$.

Fractional Brownian Motion: Let $B_\gamma(t)$, $t \geq 0$ be a standard real valued fractional Brownian motion with index $\gamma/2 \in (0,1)$. That is, $B_\gamma(t)$ is a zero-mean Gaussian process with stationary increments and covariance function

$$\mathbb{E}\, B_\gamma(t) B_\gamma(s) = \frac{1}{2}\left\{|t|^\gamma + |s|^\gamma - |t-s|^\gamma\right\}$$

Riemann-Liouville Process: Closely related to the fractional Brownian motion is the Riemann-Liouville process $W_\gamma(t)$ is defined as a fractional integration with

$$W_\gamma(t) = \frac{1}{\Gamma((\gamma+1)/2)} \int_0^t (t-s)^{(\gamma-1)/2} dB(s).$$

where $B(t)$ is a standard Brownian motion.

•Note that W_1 is just the standard Wiener process or Brownian motion and $\{W_\gamma(t)\}_{t\geq 0}$ is a self-similar zero-mean Gaussian process with scaling index $\gamma/2$, as is $B_\gamma(t)$. But $W_\gamma(t)$ does not have stationary increments and it is defined for all index $\gamma > 0$.

Theorem

Let $B(t)$ be the standard Brownian motion and $0 < \gamma < 2$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon\right)$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon / a_\gamma\right) = -C_\gamma$$

where $0 < C_\gamma < \infty$*,*

$$C_\gamma = -\inf_{\varepsilon > 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon / a_\gamma\right),$$

$$a_\gamma = \Gamma((\gamma + 1)/2) \cdot$$

$$\left(\gamma^{-1} + \int_{-\infty}^{0} ((1-s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2})^2 ds\right)^{-1/2}$$

In the Brownian motion case, i.e. $\gamma = 1$, it is well known that $C_1 = \pi^2/8$ and $a_1 = 1$.

A Relation between FBM and Riemann-Liouville Process

The relation between $W_\gamma(t)$ and $B_\gamma(t)$ becomes transparent when we write a moving average representation of $B_\gamma(t)$, $t \in \mathbb{R}$, in the form

$$B_\gamma(t) = a_\gamma \left(W_\gamma(t) + Z_\gamma(t) \right), \quad 0 \le t \le 1,$$

where

$$Z_\gamma(t) = \frac{1}{\Gamma((\gamma+1)/2)} \cdot \int_{-\infty}^{0} \{(t-s)^{(\gamma-1)/2} - (-s)^{(\gamma-1)/2}\} dB(s).$$

is a process independent of $W_\gamma(t)$.

We start with the result that for any $\beta > 0$,

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\beta} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\beta(t)| \le \varepsilon \right) = -k_\beta$$

where

$$0 < k_\beta = -\inf_{\varepsilon > 0} \varepsilon^{2/\beta} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\beta(t)| \le \varepsilon \right) < \infty.$$

•Note that the process $W_3(t)$ is the integrated Wiener process and in this case the result was first proved by using local time techniques in Khoshnevisan and Shi (1997).

The lower estimate for all $\beta > 0$,

$$\liminf_{\varepsilon \to 0} \varepsilon^{2/\beta} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\beta(t)| \le \varepsilon\right) > -\infty,$$

can be found in Li and Linde (1998) based on metric entropy connection. It can also be proved by the shift method discussed in lecture 7. When $\beta = \gamma < 2$, the estimate follows easily from

$$\mathbb{P}\left(\sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon\right) \le \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon / a_\gamma\right),$$

which is a direct consequence of the relation between $B_\gamma(t)$ and $W_\gamma(t)$, and Anderson's inequality.

Let $\hat{W}_\beta(t) = \Gamma((\beta+1)/2)W_\beta(t)$ for simplicity. Then for any $x > 0$ and $0 < \lambda < 1$, we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le 1}\left|\hat{W}_\beta(t)\right| \le x\right) \\
= \;& \mathbb{P}\left(\sup_{0\le t\le 1}\left|\int_0^t (t-s)^{(\beta-1)/2} dB(s)\right| \le x\right) \\
= \;& \mathbb{P}(\sup_{0\le t\le \lambda}\left|\hat{W}_\beta(t)\right| \le x, \ \sup_{\lambda\le t\le 1}\left|\hat{W}^*_\beta(\lambda) + \int_\lambda^t (t-s)^{(\beta-1)/2} dB(s)\right| \le
\end{aligned}$$

Note that the Gaussian processes

$$\hat{W}_\beta(t) - \hat{W}^*_\beta(\lambda) = \int_\lambda^t (t-s)^{(\beta-1)/2} dB(s), \quad \lambda \le t \le 1,$$

is independent of

$$\hat{W}_\beta(t) = \int_0^t (t-s)^{(\beta-1)/2} dB(s), \quad 0 \le t \le \lambda,$$

and $\hat{W}^*_\beta(\lambda) = \int_0^\lambda (t-s)^{(\beta-1)/2} dB(s)$. So we can use Anderson's inequality.

Thus by first conditioning on $\hat{W}_\beta(t)$, $0 \le t \le \lambda$, we obtain

$$\begin{aligned}
& \mathbb{P}\left(\sup_{0\le t\le 1}\left|\hat{W}_\beta(t)\right| \le x\right) \\
\le\ & \mathbb{P}\left(\sup_{0\le t\le \lambda}\left|\hat{W}_\beta(t)\right| \le x\right)\cdot\mathbb{P}\left(\sup_{\lambda\le t\le 1}\left|\int_\lambda^t (t-s)^{(\beta-1)/2}dB(s)\right| \le x\right) \\
=\ & \mathbb{P}\left(\sup_{0\le t\le 1}\left|\hat{W}_\beta(t)\right| \le x/\lambda^{\beta/2}\right)\cdot\mathbb{P}\left(\sup_{0\le t\le 1-\lambda}\left|\hat{W}_\beta(t)\right| \le x\right)
\end{aligned}$$

where the last equality follows from simple substitution and scaling. Taking $\lambda = 1/n$ and iterating the above procedure, we have for any $x > 0$ and any integer n

$$\mathbb{P}\left(\sup_{0\le t\le 1}\left|\hat{W}_\beta(t)\right| \le x\right) \le \mathbb{P}\left(\sup_{0\le t\le 1}\left|\hat{W}_\beta(t)\right| \le n^{\beta/2}x\right)^n$$

which finishes the proof by the scaling lemma.

It is clear from Anderson's inequality

$$\limsup_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |B_\gamma(t)| \le \varepsilon \right)$$

$$\le \lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |W_\gamma(t)| \le \varepsilon / a_\gamma \right) = -k_\gamma a_\gamma^{2/\gamma}.$$

So we only need to show the lower estimate. In particular, we need to show

Lemma

For any $0 < \gamma < 2$,

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\gamma} \log \mathbb{P}\left(\sup_{0 \le t \le 1} |Z_\gamma(t)| \le \varepsilon \right) = 0 \; .$$

Review: A lower bound on supremum

Assume $(X_t)_{t \in T}$ is a centered Gaussian process with entropy number $N(T, d; \varepsilon)$, the minimal number of balls of radius $\varepsilon > 0$, under the Dudley metric $d(s,t) = (\mathbb{E}\,|X_s - X_t|^2)^{1/2}, \quad s,t \in T$ that are necessary to cover T. Then a commonly used general lower bound estimate on the supremum is the following formulation of Talagrand:

Thm: Assume that there is a nonnegative function ψ on $\mathbb{R}_+$ such that $N(T, d; \varepsilon) \le \psi(\varepsilon)$ and such that $c_1\psi(\varepsilon) \le \psi(\varepsilon/2) \le c_2\psi(\varepsilon)$ for some constants $1 < c_1 \le c_2 < \infty$. Then, for some $C > 0$ and every $\varepsilon > 0$ we have

$$\log \mathbb{P}\left(\sup_{s,t \in T} |X_s - X_t| \le \varepsilon\right) \ge -C\psi(\varepsilon).$$

In particular, $\log \mathbb{P}\left(\sup_{t \in T} |X_t| \le \varepsilon\right) \ge -C'\psi(\varepsilon)$.

Pf of the Lemma: Note that for any $s, t \in (0,1) = T$ and $s \leq t$, with $X_\gamma(t) = \Gamma((\gamma+1)/2) Z_\gamma(t)$,

$$
\begin{aligned}
& d_\gamma^2(s,t) = \mathbb{E}\left(X_\gamma(t) - X_\gamma(s)\right)^2 \\
= \;& \mathbb{E}\left(\int_{-\infty}^{0}((t-u)^{(\gamma-1)/2} - (s-u)^{(\gamma-1)/2})dB(u)\right)^2 \\
= \;& \int_{-\infty}^{0}\left((t-u)^{(\gamma-1)/2} - (s-u)^{(\gamma-1)/2}\right)^2 du \\
= \;& \int_{s}^{\infty}\left((t-s+u)^{(\gamma-1)/2} - u^{(\gamma-1)/2}\right)^2 du.
\end{aligned}
$$

Since by the mean value theorem

$$
(t-s+u)^{(\gamma-1)/2} - u^{(\gamma-1)/2} \leq |t-s|\, u^{(\gamma-3)/2},
$$

we have for $0 < s \leq t < 1$

$$
d_\gamma(s,t) \leq (2-\gamma)^{-1/2}(t-s)\, s^{-(2-\gamma)/2}.
$$

When $s = 0$, it follows

$$d_\gamma^2(0,t) = t^\gamma \int_0^\infty \left((1+u)^{(\gamma-1)/2} - u^{(\gamma-1)/2}\right)^2 du$$

which implies $d_\gamma(0,t) \le C t^{\gamma/2}$ with $C > 0$ only depending on γ. For any $\varepsilon > 0$ small, we define numbers $0 < t_0 < t_1 < \cdots$ by $t_0 = (\varepsilon/C)^{2/\gamma}$, so that $d_\gamma(0,t_0) \le \varepsilon$, and for $i \ge 1$ by

$$(2-\gamma)^{-1/2}(t_i - t_{i-1})t_{i-1}^{-(2-\gamma)/2} = \varepsilon.$$

Let $N(\varepsilon) = \min\{n : t_n > 1\}$. Then for $1 \le i \le N(\varepsilon)$ we obtain

$$t_i = t_{i-1}(1 + (2-\gamma)^{1/2}\varepsilon t_{i-1}^{-\gamma/2}) \ge t_{i-1}(1 + (2-\gamma)^{1/2}\varepsilon),$$

thus by iterating

$$\begin{aligned} 1 \ge t_{N(\varepsilon)-1} &\ge t_0(1 + (2-\gamma)^{1/2}\varepsilon)^{N(\varepsilon)-1} \\ &= (\varepsilon/C)^{2/\gamma}(1 + (2-\gamma)^{1/2}\varepsilon)^{N(\varepsilon)-1} \end{aligned}$$

which implies $N(\varepsilon) \le c\,\varepsilon^{-1}\log(1/\varepsilon)$ for some $c > 0$. Hence using t_i, $0 \le i \le N(\varepsilon) - 1$, as centers, we finally get $N(T, d_\gamma; \varepsilon) \le N(\varepsilon) \le c\,\varepsilon^{-1}\log(1/\varepsilon)$ which finishes the proof.

Outline of the proof under L_p-norm, $p \geq 1$, i.e.

$$\lim_{\varepsilon \to 0} \varepsilon^{2/\beta} \log \mathbb{P}\left(\left(\int_0^1 |W_\beta(t)|^p \, dt\right)^{1/p} \leq \varepsilon\right) = -\kappa_{\beta,p}$$

For any $x > 0$, any $0 < \delta < 1$, and $n \geq 1$,

$$\begin{aligned}
& \mathbb{P}\left(\int_0^1 |W_\beta(t)|^p \, dt \leq x^{\beta p/2}\right) \\
\geq\ & \mathbb{P}\left(\max_{1 \leq i \leq n} \int_{(i-1)/n}^{i/n} \left|\int_0^t (t-s)^{(\beta-1)/2} dW_s\right|^p dt \leq n^{-1} x^{\beta p/2}\right) \\
\geq\ & \mathbb{P}\left(\max_{1 \leq i \leq n} \int_{(i-1)/n}^{i/n} \left|\int_0^{(i-1)/n} (t-s)^{(\beta-1)/2} dW_s\right|^p dt \leq \delta^p n^{-1} x^{\beta p/2},\right. \\
& \left. \max_{1 \leq i \leq n} \int_{(i-1)/n}^{i/n} \left|\int_{(i-1)/n}^{t} (t-s)^{(\beta-1)/2} dW_s\right|^p dt \leq (1-\delta)^p n^{-1} x^{\beta p/2}\right)
\end{aligned}$$

Using the weaker correlation inequality, we have

$$
\begin{aligned}
& \geq \quad \mathbb{P}\left(\max_{1\leq i\leq n}\int_{(i-1)/n}^{i/n}\left|\int_{(i-1)/n}^{t}(t-s)^{(\beta-1)/2}dW_s\right|^p dt\right. \\
& \qquad\qquad \left.\leq (1-\delta^2)^{p/2}(1-\delta)^p n^{-1}x^{\beta p/2}\right) \\
& \qquad \cdot\mathbb{P}\left(\max_{1\leq i\leq n}\int_{(i-1)/n}^{i/n}\left|\int_{0}^{(i-1)/n}(t-s)^{(\beta-1)/2}dW_s\right|^p dt \leq \delta^{2p}n^{-1}x^{\beta p/2}\right) \\
& = \quad P_1\cdot P_2 \quad \text{(say)}
\end{aligned}
$$

Using the weaker correlation inequality, we have

$$
\begin{aligned}
&\geq \quad \mathbb{P}\left(\max_{1\leq i\leq n}\int_{(i-1)/n}^{i/n}\left|\int_{(i-1)/n}^{t}(t-s)^{(\beta-1)/2}dW_s\right|^p dt\right.\\
&\qquad\qquad \left.\leq (1-\delta^2)^{p/2}(1-\delta)^p n^{-1}x^{\beta p/2}\right)\\
&\qquad \cdot\mathbb{P}\left(\max_{1\leq i\leq n}\int_{(i-1)/n}^{i/n}\left|\int_{0}^{(i-1)/n}(t-s)^{(\beta-1)/2}dW_s\right|^p dt \leq \delta^{2p}n^{-1}x^{\beta p/2}\right)\\
&= \quad P_1\cdot P_2 \quad \text{(say)}
\end{aligned}
$$

Next observe that for $1\leq i\leq n$

$$
\int_{(i-1)/n}^{i/n}\left|\int_{(i-1)/n}^{t}(t-s)^{(\beta-1)/2}dW_s\right|^p dt
$$

are i.i.d with the same distribution as $n^{-1-\beta p/2}\int_0^1 |W_\beta(t)|^p\,dt$. Thus

$$
P_1 = \mathbb{P}^n\left(\int_0^1 |W_\beta(t)|^p\,dt \leq (1-\delta^2)^{p/2}(1-\delta)^p(nx)^{\beta p/2}\right).
$$

Lemma

For $0 < \beta < 2$ *and any* $x > 0$ *and positive integer* n *such that* $nx < 1$,

$$
\begin{aligned}
&\mathbb{P}\left(\max_{1\le i\le n}\int_{(i-1)/n}^{i/n}\left|\int_0^{(i-1)/n}(t-s)^{(\beta-1)/2}dW_s\right| \le \delta^2 x^{\beta/2}\right) \\
\ge\ &\exp\left\{-C_{\beta,\delta}\cdot n\log(1/(nx))\right\}.
\end{aligned}
$$

Pf: This follows from entropy lower bound, after some work.

Karhunen-Loeve Expansions for Gaussian Process

Consider a centered Gaussian process $\{X_t, a \le t \le b\}$ with continuous covariance function $\sigma(s,t) = \mathbb{E}\, X_s X_t$. By Mercer's Thm, there exist eigenvalues $\lambda_n > 0$ and a complete orthonormal bases (eigenfunctions) $e_n(t)$ of

$$\lambda f(t) = \int_a^b \sigma(s,t) f(s) ds.$$

In addition, $\sigma(s,t) = \sum_{n=1}^{\infty} \lambda_n e_n(s) e_n(t)$ and the series converges absolutely and uniformly in $[a,b] \times [a,b]$. The Karhunen-Loeve expansion for X_t is $X_t = \sum_{n=1}^{\infty} \lambda_n^{1/2} \xi_n e_n(t)$ and the series converges a.e. and in $L^2(\Omega)$. We have

$$\int_a^b X_t^2 dt =^d \sum_{n=1}^{\infty} \lambda_n \xi_n^2.$$

It is difficult to find λ_n explicitly in many problems.

Exact estimates for the expansion

Theoretically, the problem has been solved in Sytaya (1974).

$$\mathbb{P}\left(\sum_{n=1}^{\infty}\lambda_n\xi_n^2 \le \varepsilon^2\right) \sim \left(4\pi\sum_{n=1}^{\infty}\left(\frac{\lambda_n\gamma_\lambda}{1+2\lambda_n\gamma_\lambda}\right)^2\right)^{-1/2} \cdot \exp\left(\varepsilon^2\gamma_\lambda - \frac{1}{2}\sum_{n=1}^{\infty}\log(1+2\lambda_n\gamma_\lambda)\right)$$

where $\gamma_\lambda = \gamma_\lambda(\varepsilon)$ is uniquely determined by

$$\varepsilon^2 = \sum_{n=1}^{\infty}\frac{\lambda_n}{1+2\lambda_n\gamma_\lambda}.$$

•Related works are given in Dudley, Hoffmann–Jorgensen and Shepp (1979), Ibragimov (1982), Zolotarev (1986), Li (1992, 1993), Dembo, Mayer-Wolf and Zeitouni (1995), Lifshits (1995), Lifshits and Linde (1996), Dunker, Lifshits, and Linde (1998), Gao, Hannig, Lee, Torcaso (2003, 2004), Nazarov (2003, 2006, 2009), Nazarov and Nikitin (2004), Gao and Li (2006), Nazarov and Pusev (2009, 2011), Pusev (2008, 2010), and many more.

A comparison theorem

•When eigenvalues λ_n can not be found explicitly, the following comparison principle started in Li (1992) under condition $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$, provides a useful computational tool. The optimal condition was proved in Gao, Hannig, Lee, Torcaso (2004). Many more refined results are known.

Theorem

If $\prod_{n=1}^{\infty}(a_n/b_n) < \infty$, *then as* $\varepsilon \to 0$

$$\mathbb{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \varepsilon^2\right) \sim \left(\prod_{n=1}^{\infty} b_n/a_n\right)^{1/2} \mathbb{P}\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \le \varepsilon^2\right).$$

For any positive integer N,

$$\log \mathbb{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \varepsilon^2\right) \sim \log \mathbb{P}\left(\sum_{n\ge N} a_n \xi_n^2 \le \varepsilon^2\right)$$

which shows that the small ball probability will not change at the logarithmic level if we delete a finite number of the terms.