Ten Lectures on Small Value Probabilities and Applications

L6: Iterating and Blocking Techniques

Wenbo V. Li

University of Delaware

http://www.math.udel.edu/~wli
wli@math.udel.edu

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We first present a vary useful iterating/blocking technique for symmetric stable processes. The lower bound is more involved since the end position of each block has to be controlled also. Then we give a block decomposition method for weighted L_p norm of Brownian motion.

Let $\{X(t) : t \geq 0\}$ be a symmetric stable process of index $\alpha \in (0, 2]$ with stationary independent increments. Define $M(t) = \sup_{0 \leq s \leq t} |X(s)|$,

Thm: Let $\rho : [0, 1] \to [0, \infty)$ be a bounded function such that $\rho(t)^\alpha$ is Riemann integrable on $[0, 1]$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\sup_{0 \leq t \leq 1} |\rho(t) X(t)| \leq \varepsilon) = -c_\alpha \int_0^1 \rho(t)^\alpha dt$$

where the constant $0 < c_\alpha < \infty$ is given by

$$c_\alpha = - \lim_{\varepsilon \to 0^+} \varepsilon^\alpha \log \mathbb{P}\Big(\sup_{0 \leq s \leq 1} |X(s)| \leq \varepsilon \Big).$$

•The existence of the limit defining c_α can be found in Mogul'skii (1974). The earlier paper Taylor (1967) obtained strictly positive, finite bounds for c_α, and there is also a variational representation of c_α in Donsker and Varadhan (1977).

•The paper by Samorodnitsky (1998) studies self-similar stable processes with stationary increments, and when they are also independent it recovers the Taylor (1967) result mentioned above. Without this independence, the upper and lower bounds in Samorodnitsky differ by a power of $\log(1/\varepsilon)$.
•The exact value of c_α is unknown for $0 < \alpha < 2$ but various estimates are available. See works of Banuelos and his coauthors.

We prove a slightly stronger result which are useful in proving Chung-Wichura type LIL.

Upper Bound: Fix sequences $\{t_i\}_{i=0}^m$, $\{a_i\}_{i=0}^m$, and $\{b_i\}_{i=0}^m$ such that $0 = t_0 < t_1 < \cdots < t_m$ and
$a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$. Then

$$\limsup_{\varepsilon \to 0^+} \varepsilon^\alpha \log \mathbb{P}(a_i \varepsilon \leq M(t_i) \leq b_i \varepsilon, 1 \leq i \leq m) \leq -c_\alpha \sum_{i=1}^m (t_i - t_{i-1})/b_i^\alpha.$$

Pf:. Let $A_i = \left\{ \sup_{t_{i-1} \leq s < t_i} |X(s)| \leq b_i \varepsilon \right\}$ for $i = 1, \ldots, m$. Then it is easy to see

$$\mathbb{P}(a_i \varepsilon \leq M(t_i) \leq b_i \varepsilon, \ 1 \leq i \leq m) \leq P\big(\bigcap_{i=1}^m A_i \big).$$

Furthermore, we have

$$
\begin{aligned}
\mathbb{P}\big(\bigcap_{i=1}^{m} A_i\big) &= \int_{\mathbb{R}} \mathbb{P}\big(\bigcap_{i=1}^{m-1} A_i, \sup_{t_{m-1}\le s<t_m} |X(s)-X(t_{m-1})+x| \le b_m\varepsilon \Big| \\
&\qquad X(t_{m-1})=x\big) d\mathbb{P}_{X(t_{m-1})}(x) \\
&= \int_{\mathbb{R}} \mathbb{P}\big(\sup_{t_{m-1}\le s<t_m} |X(s)-X(t_{m-1})+x| \le b_m\varepsilon\big) \cdot \\
&\qquad \cdot \mathbb{P}\big(\bigcap_{i=1}^{m-1} A_i \Big| X(t_{m-1})=x\big) d\mathbb{P}_{X(t_{m-1})}(x),
\end{aligned}
$$

since $\sup_{t_{m-1}\le s<t_m} |X(s)-X(t_{m-1})+x|$ is independent of $X(t_{m-1})$ and $\bigcap_{i=1}^{m-1} A_i$ by the independent increments property of $X(t)$.

By Anderson's inequality for independent clock representation $X(t) = W(S_{\alpha/2}(t))$,

$$
\begin{aligned}
& \mathbb{P}\big(\sup_{t_{m-1} \le s < t_m} |X(s) - X(t_{m-1}) + x| \le b_m \varepsilon\big) \\
\le \; & \mathbb{P}\big(\sup_{t_{m-1} \le s < t_m} |X(s) - X(t_{m-1})| \le b_m \varepsilon\big) \\
= \; & \mathbb{P}\big(\sup_{0 \le s \le 1} |X(s)| \le b_m \varepsilon / (t_m - t_{m-1})^{1/\alpha}\big),
\end{aligned}
$$

where the equality follows from the scaling property of $\{X(t) : t \ge 0\}$ and the homogeneity of the increments. Thus

$$
\mathbb{P}\big(\bigcap_{i=1}^{m} A_i\big) \le \; P\big(\bigcap_{i=1}^{m-1} A_i\big) \cdot \mathbb{P}\big(\sup_{0 \le s \le 1} |X(s)| \le b_m \varepsilon / (t_m - t_{m-1})^{1/\alpha}\big),
$$

and iterating the above estimate implies the result.

To obtain a reverse estimate, we need the following lemma.
Lemma: Given $\delta > 0$,

$$\lim_{\varepsilon\to 0^+} \varepsilon^\alpha \log \mathbb{P}(M(1) \le \varepsilon,\ |X(1)| \le \varepsilon\delta) = -c_\alpha.$$

•We have for given positive numbers $a < b$ and $\delta > 0$,

$$\lim_{\varepsilon\to 0^+} \varepsilon^\alpha \log \mathbb{P}(a\varepsilon \le M(1) \le b\varepsilon,\ |X(1)| \le \varepsilon\delta) = -c_\alpha/b^\alpha.$$

Pf: We assume $\delta \in (0,1)$, Note that for any $\theta \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(M(1) \le \varepsilon, |X(1)| \le \varepsilon\delta) &\ge \mathbb{P}(M(1) < \varepsilon,\ |X(1)+\theta| \le \varepsilon\delta) \\ &= \mathbb{P}(M(1) \le \varepsilon,\ |X(1)+\theta| \le \varepsilon\delta). \end{aligned}$$

where the inequality is due to Anderson's inequality applied conditionally to the Gaussian probability in only the last coordinate. Thus

$$\begin{aligned} \mathbb{P}(M(1) \le \varepsilon) &\le \sum_{j=-[1/\delta]}^{[1/\delta]} \mathbb{P}(M(1) \le \varepsilon,\ |X(1)+j\varepsilon\delta| \le \varepsilon\delta) \\ &\le \Big(2\Big[1/\delta\Big]+1\Big)\mathbb{P}(M(1) \le \varepsilon,\ |X(1)| \le \varepsilon\delta). \end{aligned}$$

Lower Bound: Fix sequences $\{t_i\}_{i=0}^m$, $\{a_i\}_{i=0}^m$, $\{b_i\}_{i=0}^m$ such that $0 = t_0 < t_1 < \cdots < t_m$ and $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m$. Then, for every $\gamma > 0$,

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \varepsilon^\alpha \log P(a_i\varepsilon \le M(t_i) \le b_i\varepsilon, 1 \le i \le m, |X(t_m)| \le b_m\gamma\varepsilon) \\ \ge\ & -c_\alpha \sum_{i=1}^m \frac{t_i - t_{i-1}}{b_i^\alpha}. \end{aligned}$$

Pf: Take a small $\delta > 0$ such that $\delta < \gamma$ and $a_i(1+\delta) < b_i(1-\delta)$ for all $1 \le i \le m$. Define

$$B_i = \left\{ a_i\varepsilon \le \sup_{t_{i-1}\le s\le t_i} |X(s)| \le b_i\varepsilon,\ |X(t_i)| \le b_i\delta\varepsilon \right\}$$

for $i = 1, \ldots, m$. Then

$$\{a_i\varepsilon \le M(t_i) \le b_i\varepsilon,\ 1 \le i \le m,\ |X(t_m)| \le b_m\gamma\varepsilon\} \supseteq \bigcap_{i=1}^m B_i.$$

On the other hand, if for $i = 1, \dots, m$.

$$\begin{aligned} A_i &= \Big\{ a_i(1+\delta)\varepsilon \le \sup_{t_{i-1}\le s\le t_i} |X(s) - X(t_{i-1})| \le b_i(1-\delta)\varepsilon, \\ &\qquad |X(t_i) - X(t_{i-1})| \le (b_i - b_{i-1})\delta\varepsilon \Big\} \end{aligned}$$

then

$$\mathbb{P}\Big(\bigcap_{i=1}^{m} B_i \Big) \ge \mathbb{P}\Big(\bigcap_{i=1}^{m-1} B_i \cap A_m \Big) = \mathbb{P}\Big(\bigcap_{i=1}^{m-1} B_i \Big) \cdot \mathbb{P}(A_m) \ge \prod_{i=1}^{m} \mathbb{P}(A_i).$$

and $\mathbb{P}(A_i) = \mathbb{P}\Big(\frac{a_i(1+\delta)\varepsilon}{(t_i - t_{i-1})^{1/\alpha}} \le M(1) \le \frac{b_i(1-\delta)\varepsilon}{(t_i - t_{i-1})^{1/\alpha}},\ |X(1)| \le \frac{(b_i - b_{i-1})\delta\varepsilon}{(t_i - t_{i-1})^{1/\alpha}} \Big)$. Recall

$$B_i = \left\{ a_i\varepsilon \le \sup_{t_{i-1}\le s\le t_i} |X(s)| \le b_i\varepsilon,\ |X(t_i)| \le b_i\delta\varepsilon \right\}.$$

Weighted L_p-norm for BM

Thm: Under regularity conditions on $\rho : [0, \infty) \to [0, \infty]$ we have for $1 \leq p < \infty$

$$\begin{aligned} & \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left(\left(\int_0^\infty |\rho(t) W(t)|^p dt \right)^{1/p} \leq \varepsilon \right) \\ = & \; -\kappa_p \left(\int_0^\infty \rho(t)^{2p/(2+p)} dt \right)^{(2+p)/p} \end{aligned}$$

where κ_p is the small ball constant for BM under L_p-norm, see lecture 2.

•The full generality of this result is given in Li (2001) and related works can be found in Mogul'skii (1974), Shi (1996), Berthet and Shi (1999), Li (1998), Lifshits and Linde (2000).

Let $\{X(t); 0 \leq t \leq 1\}$ be a real–valued continuous Gaussian Markov process with mean zero and covariance $\sigma(s,t) = EX(s)X(t) \neq 0$ for $0 < s, t < 1$. It is known that we can write $\sigma(s,t) = G(\min(s,t))H(\max(s,t))$ with $G > 0, H > 0$ and G/H nondecreasing on the interval $(0,1)$.

Cor:

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\|X(t)\|_p < \varepsilon) = -\kappa_p \left(\int_0^1 (G'H - H'G)^{p/(2+p)} dt \right)^{(2+p)/p}.$$

•We have the following representation for Gaussian Markov processes

$$X(t) = h(t)W(g(t))$$

with $g(t)G(t)/H(t) > 0$ nondecreasing on the interval $(0,1)$ and $h(t) = H(t) > 0$ on the interval $(0,1)$.

Ex: Let $U(t)$ be the O-U process with $\mathbb{E}\, U(s)U(t) = \sigma^2 e^{-\theta|t-s|}$ for $\theta > 0$ and any $s, t \in [a, b]$, $-\infty < a < b < \infty$. The we have for $1 \le p \le \infty$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\|U(t)\|_p \le \varepsilon \right) = -2\sigma^2 \theta (b-a)^{(2+p)/p} \kappa_p.$$

Ex: For $\alpha < (2+p)/2p$, $p \ge 1$,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^1 |t^{-\alpha} W(t)|^p dt \le \varepsilon^p \right) = -\kappa_p \left(\frac{2+p}{2+p-2\alpha p} \right)^{(2+p)/p}.$$

Thus by the exponential Tauberian theorem and scaling property of Brownian motion,

$$\lim_{t \to \infty} t^{-(2+p-2\beta)/(2+p)} \log \mathbb{E} \exp\left\{ -\lambda \int_0^t \frac{|W(s)|^p}{s^\beta} ds \right\} = -\frac{2+p}{2+p-2\beta} \lambda_1(p) \lambda^{2/(2+p)}$$

for $\beta < (2+p)/2$ and $\lambda > 0$.

Our proof is given in three steps. First we assume that $\rho(t)$ is bounded and $\rho(t)^{2p/(2+p)}$ Riemann integrable on $[0, T]$ and $\rho(t) = 0$ for $t \geq T$. In the second step, we assume $\rho(t)$ is non-increasing on $[0, a]$ for some $a > 0$ small and $\rho(t) = 0$ for $t \geq T$. The weaker Gaussian correlation inequality can be used but it is not critical here since we can form independent increment by introducing the value at $t = a$. In the last step, we show the theorem over the whole positive half line. In this step, the weaker Gaussian correlation inequality is important.

•We only show the first step in this lecture to illustrate the blocking technique.

Prop: Let $\rho : [0, T] \to [0, \infty)$ be a bounded function on $[0, T]$, $0 < T < \infty$. Then

$$\begin{aligned} & \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left(\int_0^T |\rho(t) W(t)|^p dt \le \varepsilon^p \right) \\ \ge \quad & -\kappa_p \inf \left(\sum_{i=1}^n M_i^{2p/(2+p)} (t_i - t_{i-1}) \right)^{(2+p)/p} \end{aligned}$$

and

$$\begin{aligned} & \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left(\int_0^T |\rho(t) W(t)|^p dt \le \varepsilon^p \right) \\ \le \quad & -\kappa_p \sup \left(\sum_{i=1}^n m_i^{2p/(2+p)} (t_i - t_{i-1}) \right)^{(2+p)/p} . \end{aligned}$$

where the infimum and supremum being taken over all partitions $\mathcal{P} = \{t_i\}_0^n$ and

$$m_i = \inf_{t_{i-1} \le t \le t_i} \rho(t) \quad \text{and} \quad M_i = \sup_{t_{i-1} \le t \le t_i} \rho(t).$$

In particular, if $\rho(t)^{2p/(2+p)}$ is Riemann integrable, then

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p dt \le \varepsilon^p\right) \\
= \quad & -\kappa_p \left(\int_0^T \rho(t)^{2p/(2+p)} dt\right)^{(2+p)/p} .
\end{aligned}$$

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Stochastic Representation for BM

Fix a finite partition $\mathcal{P} = \{t_i\}_0^n$ of $[0, T]$ such that

$$0 = t_0 < t_1 < \cdots < t_n = T.$$

Let $B_1(t), B_2(t), \cdots B_n(t)$, $0 \le t \le 1$ be independent standard Brownian bridges that are also independent of $W(t)$. Define for $t_{i-1} \le t \le t_i$

$$\begin{aligned} \widehat{W}(t) &= W(t_{i-1}) + (W(t_i) - W(t_{i-1})) \frac{t - t_{i-1}}{t_i - t_{i-1}} + \\ &\quad + \sqrt{t_i - t_{i-1}} B_i \left(\frac{t - t_{i-1}}{t_i - t_{i-1}} \right) \end{aligned}$$

Then it is easy to check that $\left\{ \widehat{W}(t), 0 \le t \le T \right\}$ is a standard Brownian motion by checking covariance function.

•From the weaker correlation inequality,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\|W\|_p \le \varepsilon \right) = \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\|B\|_p \le \varepsilon \right) = -\kappa_p$$

since $B(t) = W(t) - tW1)$, $0 \le t \le 1$.

Upper Bound

We have

$$
\begin{aligned}
& \mathbb{P}\left(\left(\int_0^T |\rho(t)W(t)|^p dt\right)^{1/p} \leq \varepsilon\right) \\
= \; & \mathbb{P}\left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \rho^p(t)|\widehat{W}(t)|^p dt \leq \varepsilon^p\right) \\
\leq \; & \mathbb{P}\left(\sum_{i=1}^n m_i^p \int_{t_{i-1}}^{t_i} |\widehat{W}(t)|^p dt \leq \varepsilon^p\right) \\
= \; & \mathbb{P}\left(\sum_{i=1}^n m_i^p (t_i - t_{i-1}) \int_0^1 |(1-t)W(t_{i-1}) + tW(t_i)\right. \\
& \qquad \left. + \sqrt{t_i - t_{i-1}} B_i(t)|^p dt \leq \varepsilon^p\right) \\
\leq \; & \mathbb{P}\left(\sum_{i=1}^n m_i^p (t_i - t_{i-1})^{1+p/2} \int_0^1 |B_i(t)|^p dt \leq \varepsilon^p\right)
\end{aligned}
$$

where the second inequality follows from Anderson's inequality and the fact that $B_i(t)$, $1 \leq i \leq n$, are independent of $W(t)$.

Thus from lecture 2 or 3 on independent sums,

$$
\begin{aligned}
& \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\left(\int_0^T |\rho(t)W(t)|^p dt\right)^{1/p} \le \varepsilon\right) \\
\le \; & \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sum_{i=1}^{n} m_i^p (t_i - t_{i-1})^{1+p/2} \int_0^1 |B_i(t)|^p dt \le \varepsilon^p\right) \\
= \; & -\kappa_p \left(\sum_{i=1}^{n} m_i^{2p/(2+p)} (t_i - t_{i-1})\right)^{(2+p)/p} .
\end{aligned}
$$

Lower Bound

By the representation for BM used for the upper bound, we have

$$
\begin{aligned}
& \mathbb{P}\left(\int_0^T |\rho(t)W(t)|^p dt \le \varepsilon^p\right) \\
\ge\ & \mathbb{P}\left(\sum_{i=1}^n M_i^p (t_i - t_{i-1}) \int_0^1 |(1-t)W(t_{i-1}) + tW(t_i) \right. \\
& \qquad \left. + \sqrt{t_i - t_{i-1}} B_i(t)|^p dt \le \varepsilon^p \right) \\
\ge\ & \mathbb{P}\left(\int_0^1 |(1-t)W(t_{i-1}) + tW(t_i) + \sqrt{t_i - t_{i-1}} B_i(t)|^p dt \le Q_i^p \varepsilon^p, \right. \\
& \qquad \left. 1 \le i \le n \right), \qquad \text{for} \quad \sum M_i^p (t_i - t_{i-1}) Q_i^p = 1 \\
\ge\ & \mathbb{P}\left(\int_0^1 |(1-t)W(t_{i-1}) + tW(t_i)|^p dt \le \delta^p \varepsilon^p, \right. \\
& \qquad \left. \int_0^1 |\sqrt{t_i - t_{i-1}} B_i(t)|^p dt \le (Q_i - \delta)^p \varepsilon^p, 1 \le i \le n \right) \\
\ge\ & \mathbb{P}\left(|W(t_i)| \le \delta\varepsilon, \int_0^1 |\sqrt{t_i - t_{i-1}} B_i(t)|^p dt \le (Q_i - \delta)^p \varepsilon^p, 1 \le i \le n\right.
\end{aligned}
$$

For the last line, we have

$$
\begin{aligned}
& \mathbb{P}\left(|W(t_i)| \le \delta\varepsilon, \int_0^1 |\sqrt{t_i - t_{i-1}} B_i(t)|^p dt \le (Q_i - \delta)^p \varepsilon^p, 1 \le i \le n \right. \\
= \; & \mathbb{P}\left(\max_{1 \le i \le n} |W(t_i)| \le \delta\varepsilon \right) \cdot \\
& \cdot \prod_{i=1}^n \mathbb{P}\left(\int_0^1 |B_i(t)|^p dt \le (t_i - t_{i-1})^{-p/2} (Q_i - \delta)^p \varepsilon^p \right)
\end{aligned}
$$

where the equality follows from the independence of $B_i(t)$, $1 \le i \le n$ and $W(t)$. Thus

$$
\begin{aligned}
& \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^T |\rho(t) W(t)|^p dt \le \varepsilon^p \right) \\
\ge \; & \sum_{i=1}^n \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\int_0^1 |B_i(t)|^p dt \le (t_i - t_{i-1})^{-p/2} (Q_i - \delta)^p \varepsilon^p \right) \\
= \; & -\kappa_p \sum_{i=1}^n (Q_i - \delta)^{-2} (t_i - t_{i-1}).
\end{aligned}
$$

Taking $\delta \to 0$ and we need Q_i to match the upper bound.

This forces

$$Q_i = M_i^{-p/(2+p)} \left(\sum_{i=1}^{n} M_i^{2p/(2+p)} (t_i - t_{i-1}) \right)^{-1/p} > 0, \quad 1 \leq i \leq n.$$

and we can pick $0 < \delta < \min_{1 \leq i \leq n} Q_i$.

$\bullet$Note that

$$\sum_{i=1}^{n} M_i^p (t_i - t_{i-1}) Q_i^p = 1.$$