Ten Lectures on Small Value Probabilities and Applications

L4: Applications of Weak Gaussian Correlation Inequality

Wenbo V. Li

University of Delaware

http://www.math.udel.edu/~wli
wli@math.udel.edu

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We treat the sum of two *not* necessarily independent Gaussian random vectors in a separable Banach space. The main ingredients are Anderson's inequality and the weaker correlation inequality. Various applicants are provided to show the power of the method.

Abstract: Gaussian inequalities play a fundamental role in the study of high dimensional probability and stochastic analysis. We will first provide an overview of various Gaussian inequalities and then present several recent results and conjectures for Gaussian measure/vectors, together with various applications.
Gaussian inequalities, whose goal, loosely speaking, consists of searching for an inequality between dependent (complicated) and independent (simpler) structures that becomes an equality in certain (possibility limiting) cases.

Some literature

•Ledoux, M. and Talagrand, M. (1991). *Probability on Banach Spaces*, Springer, Berlin.
•Lifshits, M.A. (1995). *Gaussian Random Functions*. Kluwer Academic Publishers, Boston.
•Yurinsky, V. (1995). Sums and Gaussian Vectors, *Lecture Notes in Math.* **1617**, Springer-Verlag.
•Ledoux, M. (1996). Isoperimetry and Gaussian Analysis, *Lectures on Probability Theory and Statistics, Lecture Notes in Math.* **1648** 165–294, Springer-Verlag.
•V.I. Bogachev, "Gaussian measures", AMS 1998.
•Ledoux, M. (1999). Concentration of measure and logarithmic Sobolev inequalities, *Lecture Notes in Mathematics* **1709**, 120-216.
•Li, W.V. and Shao, Q-M. (2001). Gaussian processes: inequalities, small ball probabilities and applications. Handbook of Statistics, Vol. **19**, *Stochastic processes: Theory and methods*, 533-598, Elsevier, New York.
•Talagrand, M. (2003). *Spin Glasses: a Challenge to Mathematicians*, Cavity and Mean Field Models. A Series of

Fundamental Gaussian Inequalities

- Isoperimetric inequalities: Gaussian Isoperimetric inequalities; Ehrhard's inequality; Shift inequalities; S-inequality; B-inequality; etc.
- Comparison inequalities: Anderson's inequality; Slepian's inequality; Gordon's min-max inequalities; Reverse Slepian type inequalities; etc.
- Correlation inequalities and conjectures: Sidak inequality; Weak correlation inequality; etc.
- Concentration and deviation inequalities: Dudley, Fernique, Berman, Talagrand, etc.
- Functional inequalities: Poincare inequalities; Logarithmic Sobolev inequality; Transportation-entropy-information inequalities; etc.

The density and distribution function of the standard Gaussian (normal) distribution on the real line $\mathbb{R}$ are

$$\phi(x) = (2\pi)^{-1/2} \exp\{-x^2/2\} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(t)dt.$$

Let γ_n denote the canonical Gaussian measure on $\mathbb{R}^n$ with density function

$$\phi_n(x) = (2\pi)^{-n/2} \exp(-|x|^2/2)$$

with respect to Lebesgue measure, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$. All results for γ_n on $\mathbb{R}^n$ can be used to determine the appropriate infinite dimensional analogue by a classic approximation argument.

The n-dimensional normal density is

$$\begin{aligned} f(x) &= f(x_1, \cdots, x_n) \\ &= \frac{1}{(2\pi)^{n/2}(\det(A))^{1/2}} \exp\left\{-\frac{1}{2}\langle x, A^{-1}x^T\rangle\right\} \end{aligned}$$

where the covariance $A = (\mathbb{E}\, X_i X_j)$ is an $n \times n$ positive definite symmetric matrix.

Gaussian Random Element in Banach Space

A centered random element X in a separable Banach space E is called Gaussian if $f(X)$ is mean-zero Gaussian for any $f \in E^*$, and equivalently, the characteristic functional of X is

$$\hat{\mu}(f) := \mathbb{E}\, e^{i\langle f, X\rangle} = \exp(-\frac{1}{2}\langle f, Kf\rangle$$

for some positive definite operator $K : E^* \to E$.
Ex: For $E = \mathbb{R}^n$, $E^* = \mathbb{R}^n$ and thus $X = (X_1, \cdots, X_n)$ is a centered Gaussian if

$$a_1 X_1 + \cdots + a_n X_n$$

is a one-dim Gaussian r.v.
•Uncorrelated Gaussian vectors are independent.

Gaussian Process, Operator and RKHS

The following statements are equivalent:

(i). X is a centered Gaussian random vector with law $\mu = \mathcal{L}(X)$ in a separable Banach space E.

(ii). There exist a separable Hilbert space H and an operator $u : H \to E$ such that $\sum_{j=1}^{\infty} \xi_j u(f_j)$ converges a.s. in E for one (each) ONB $(f_j)_{j=1}^{\infty}$ in H and

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j u(f_j)$$

where ξ_j are i.i.d. $N(0, 1)$.

(iii). There are $x_1, x_2, \ldots$ in E such that $\sum_{j=1}^{\infty} \xi_j x_j$ converges a.s. in E and $X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j x_j$.

- The series $\sum_{j=1}^{\infty} \xi_j u(f_j)$ converges a.s. implies that u is compact and the RKHS $H_\mu = u(H)$ with compact unit ball K_μ in E.
- The RKHS H_μ can also be described as the completion of the range of the mapping $S : E^* \to E$ defined by the Bochner integral $Sf = \int_E x f(x) d\mu(x), f \in E^*$.

Isoperimetry

The main geometric property of both measures (Lebesgue and Gaussian) is an isoperimetric inequality. For Lebesgue measure it is classical (J. Steiner 1842, H. Schwarz 1884). Among all bodies of a given volume, a ball minimizes the surface area. For Gaussian measure, an isoperimetric inequality has appeared independently in the work

- V.N. Sudakov, B.S. Tsirelson, "Extremal properties of half-spaces for spherically invariant measures", Zapiski LOMI **41** (1974), 14-24 (Russian); J. Soviet Mathematics **9**, (1978), 9-18 (English).
- C. Borell, "The Brunn-Minkowski inequality in Gauss space", Invent. Math. **30** (1975), 207-216.

Statement: Among all sets of a given Gaussian measure, a half-space minimizes the Gaussian measure of a neighborhood.

- The usefulness for the theory of Gaussian processes was only realized in the mid-80's. Many (better) proofs have appeared later, see A. Ehrhard (1983), M. Ledoux (1994), S. Bobkov (1997), C. Borell (2005).

The Gaussian Isoperimetric Inequality

Let A be a measurable subset of $\mathbb{R}^n$ such that

$$\gamma(A) = \gamma(H) = \Phi(a) = \int_{-\infty}^{a} \phi(t)dt,$$

where H is a half–space $\{x \in \mathbb{R}^n : (x, u) < a\}$, for some $u \in \mathbb{R}^n$ with $|u| = 1$ and $a \in [-\infty, \infty]$. Then, for every $r \geq 0$

$$\gamma(A + rU) \geq \gamma(H + rU) = \Phi(a + r) = \Phi(\Phi^{-1}(\gamma(A)) + r)$$

where U is the open unit ball in $\mathbb{R}^n$ and
$A + rU = \{a + ru : a \in A, u \in U\}$.

• Ehrhard's inequality: For any Borel sets A and B of $\mathbb{R}^n$, and $0 \leq \lambda \leq 1$,

$$\begin{aligned} & \Phi^{-1} \circ \gamma_n(\lambda A + (1-\lambda)B) \\ \geq \ & \lambda \Phi^{-1} \circ \gamma_n(A) + (1-\lambda)\Phi^{-1} \circ \mu(B). \end{aligned}$$

Open: Bounds on $\gamma_n(A + rU)$ for symmetric and convex set A.

• Concentration and deviation inequalities

Let f be Lipschitz function on $\mathbb{R}^n$ with

$$\|f\|_{Lip} = \sup\{|f(x) - f(y)|/|x - y| : x, y \in \mathbb{R}^n\}.$$

Denote further by M_f a median of f for μ and by $\mathbb{E}\, f = \int f d\mu(x)$ for the expectation of f. Then

$$\mu(|f - M_f| > t) \le \exp\{-t^2/2\,\|f\|^2_{Lip}\}$$

and

$$\mu(|f - \mathbb{E}\, f| > t) \le 2\exp\{-t^2/2\,\|f\|^2_{Lip}\}$$

•Another version of the above result can be stated as follows. Let $\{X_t, t \in T\}$ be a centered Gaussian process with

$$d(s,t) = (\mathbb{E}\,|X_s - X_t|^2)^{1/2}, \quad s, t \in T$$

and $\sigma^2 = \sup_{t \in T} \mathbb{E}\, X_t^2$. Then for all $x > 0$, we have

$$\mathbb{P}\Big(\sup_{t\in T} X_t - \mathbb{E}\,\sup_{t \in T} X_t \ge x\Big) \le \exp\Big(-\frac{x^2}{2\sigma^2}\Big).$$

There are several other inequalities of various flavor given by Dudley (1967), Fernique (1972) and Berman (1985), etc.

Comparison inequalities

Slepian's lemma: If $\mathbb{E}\, X_i^2 = \mathbb{E}\, Y_i^2$ and $\mathbb{E}\, X_i X_j \leq \mathbb{E}\, Y_i Y_j$ for all $i, j = 1, 2, ..., n$, then for any x,

$$\mathbb{P}\left(\max_{1 \leq i \leq n} X_i \leq x\right) \leq \mathbb{P}\left(\max_{1 \leq i \leq n} Y_i \leq x\right).$$

•The basic comparison identity: Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be a function with bounded second derivatives. Then for centered Gaussian random vectors $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$,

$$\mathbb{E}\, f(X) - \mathbb{E}\, f(Y) = \frac{1}{2} \int_0^1 \sum_{1 \leq i,j \leq n} (\mathbb{E}\, X_i X_j - \mathbb{E}\, Y_i Y_j)$$
$$\cdot \mathbb{E}\, \frac{\partial^2 f}{\partial x_i \partial x_j}((1-\lambda)^{1/2} X + \lambda^{1/2} Y) d\lambda.$$

•Other interesting and useful extensions of Slepian's inequality, involving min-max, etc, can be found in Gordon (1985).

Reverse Slepian type Inequality

A very useful extension of Slepian's inequality in the 'reverse' direction is established in Li and Shao (2002) with three important applications: (i). A conjecture of Erdős and Révész (1990) is settled; (ii). A conjecture of Kesten (1992) is confirmed. (ii). Sharper estimate on the positivity exponent ($b > 0.5$) for random polynomials of even degree.

Thm: Let $\{\xi_i, 1 \le i \le n\}$ and $\{\eta_i, 1 \le i \le n\}$ be two normal random vectors with mean zero and variance one. Assume that $\mathbb{E}\,\xi_i \xi_j \ge \mathbb{E}\,\eta_i \eta_j \ge 0$ for $1 \le i, j \le n$. Then for $u \ge 0$

$$\begin{aligned}\mathbb{P}\left(\max_{1\le i\le n} \eta_i \le u\right) \;&\le\; \mathbb{P}\left(\max_{1\le i\le n} \xi_i \le u\right) \le \mathbb{P}\left(\max_{1\le i\le n} \eta_i \le u\right) \cdot \\ &\cdot \prod_{1\le i<j\le n} \left(\frac{\pi - 2\arcsin(\mathbb{E}\,\eta_i\eta_j)}{\pi - 2\arcsin(\mathbb{E}\,\xi_i\xi_j)}\right)^{\exp\{-u^2/(1+\mathbb{E}\,\xi_i\xi_j)\}}\end{aligned}$$

•There are other related variations under slightly different conditions.

Gaussian Inequalities for Products

Thm: For any centered joint Gaussian random variables $X_1, \cdots, X_n$ with the covariance matrix Σ,

$$\mathbb{E}\,|X_1 X_2 \cdots X_n| \le \sqrt{\mathrm{perm}(\Sigma)} \le (\mathbb{E}\,X_1^2 X_2^2 \cdots X_n^2)^{1/2}.$$

The first inequality is due to the speaker, see Li and Wei (2009) and the second inequality is due to Frenkel (2008).
Gaussian Products Conjecture: For any centered Gaussian vector $(X_1, \cdots, X_n)$ and positive integer m,

$$\mathbb{E}\,X_1^{2m} \cdots X_n^{2m} \ge \mathbb{E}\,X_1^{2m} \cdots \mathbb{E}\,X_n^{2m}.$$

•Frenkel (2008): Yes for $m = 1$. In fact,

$$\mathbb{E}\,X_1^2 \cdots X_n^2 \ge \mathrm{perm}(\Sigma) \ge \mathbb{E}\,X_1^2 \cdots \mathbb{E}\,X_n^2.$$

via hafnians and exterior algebra for the first inequality above.

•The B-inequality in Cordero-Erausquin, Fradelizi and Maurey (2005):

$$\mu(aK) \cdot \mu(bK) \leq \mu^2(\sqrt{ab}K)$$

for any symmetric convex set K and positive numbers a and b.
•Equivalently, for any norm $\|\cdot\| = \|\cdot\|_K$ on $\mathbb{R}^n$ or a Banach space, and centered Gaussian random vector X,

$$\mathbb{P}(\|X\| \leq a) \cdot \mathbb{P}(\|X\| \leq b) \leq \mathbb{P}^2(\|X\| \leq \sqrt{ab}).$$

•Relation between K and $\|x\| = \|\cdot\|_K$:

$$\|x\| = \inf\{t > 0 : x \in tK\}, \quad K = \{x : \|x\| \leq 1\}.$$

Shift Inequalities

•Anderson's inequality: For every convex symmetric set K,

$$\mu(K + x) \le \mu(K)$$

which follows easily from log-concavity of Gaussian measure.
•Measure of shifted balls: For any $f \in H_\mu$ and $r > 0$,

$$\exp\{-|f|_\mu^2/2\} \cdot \mu(x : \|x\| \le r) \le \mu(x : \|x - f\| \le r) \le \mu(x : \|x\| \le r).$$

Furthermore, as $\varepsilon \to 0$,

$$\mathbb{P}(\|X - f\| \le \varepsilon) \sim \exp\{-|f|_{\mu(X)}^2/2\} \cdot \mathbb{P}(\|X\| \le \varepsilon).$$

The upper bound follows from Anderson's inequality. The lower bound follows from the Cameron-Martin formula. Various sharp refinements and applications are given in Kuelbs, Li and Talagrand (1994), Kuelbs, Li and Linde (1994), and Kuelbs and Li (1998).

Correlation inequalities

The Gaussian correlation conjecture: For any two symmetric convex sets A and B in a separable Banach space E and for any centered Gaussian measure μ on E,

$$\mu(A \cap B) \geq \mu(A)\mu(B).$$

An equivalent formulation: If $(X_1, \ldots, X_n)$ is a centered, Gaussian random vector, then

$$\mathbb{P}\left(\max_{1\leq i\leq n} |X_i| \leq 1\right) \geq \mathbb{P}\left(\max_{1\leq i\leq k} |X_i| \leq 1\right) \mathbb{P}\left(\max_{k+1\leq i\leq n} |X_i| \leq 1\right)$$

for each $1 \leq k < n$.

- Sidak inequality: The above holes for $k = 1$ or any slab B.

The weaker Correlation inequality:

For any $0 < \lambda < 1$, any symmetric, convex sets A and B,

$$\mu(A \cap B)\mu(\lambda^2 A + (1-\lambda^2)B) \geq \mu(\lambda A)\mu((1-\lambda^2)^{1/2}B).$$

In particular,

$$\mu(A \cap B) \geq \mu(\lambda A)\mu((1-\lambda^2)^{1/2}B)$$

and

$$\mathbb{P}(X \in A, Y \in B) \geq \mathbb{P}\Big(X \in \lambda A\Big)\mathbb{P}\Big(Y \in (1-\lambda^2)^{1/2}B\Big)$$

for any centered joint Gaussian vectors X and Y.
The varying parameter λ plays a fundamental role in applications, see Li (1999). It allows us to justify

$$\mu(A \cap B) \approx \mu(A) \ \text{ if } \ \mu(A) \ll \mu(B).$$

Note also that

$$\mu(\cap_{i=1}^m A_i) \geq \prod_{i=1}^m \mu(\lambda_i A_i)$$

for any $\lambda_i \geq 0$ with $\sum_{i=1}^m \lambda_i^2 = 1$.

The proof in Li (1999) follows ideas in Schechtman, Schlumprecht and Zinn (1998), where the case $\lambda = 1/\sqrt{2}$ was proved. It is based on the rotational invariance of the measure $\mu_n \times \mu_n$ for $(x, y) \mapsto (\lambda x + \eta y, \eta x - \lambda y)$ and the following theorem on log concave functions.

Theorem (Prékopa '72, Leindler '73). If f is log-concave on $\mathbb{R}^n$ and $1 \leq k < n$, then the function $g : \mathbb{R}^k \to \mathbb{R}$, with

$$g(x_1, \ldots, x_k) = \int_{\mathbb{R}^{n-k}} f(x_1, \ldots, x_k, z_1, \ldots, z_{n-k})\, dz$$

is also log concave.

Corollary. If f and g are log concave, so is $y \mapsto \int f(x+y)g(x)\, dx$.

For the weaker correlation inequality established in Li (1999), here is a very simple proof given in Li and Shao (2001). Let $a = (1-\lambda^2)^{1/2}/\lambda$, and (X^*, Y^*) be an independent copy of (X, Y). Then $X - aX^*$ and $Y + Y^*/a$ are independent. Thus, by Anderson inequality

$$\begin{aligned} \mathbb{P}(X \in A, Y \in B) &\geq \mathbb{P}(X - aX^* \in A, Y + Y^*/a \in B) \\ &= \mathbb{P}(X - aX^* \in A)\mathbb{P}(Y + Y^*/a \in B) \\ &= \mathbb{P}\Big(X \in \lambda A\Big)\mathbb{P}\Big(Y \in (1-\lambda^2)^{1/2} B\Big). \end{aligned}$$

Consider the sums of two centered Gaussian random vectors X and Y in a separable Banach space E with norm $\|\cdot\|$.

Thm: If X and Y are independent and

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}(\|X\| \le \varepsilon) = -C_X, \quad \lim_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}(\|Y\| \le \varepsilon) = -C_Y$$

with $0 < \gamma < \infty$ and $0 \le C_X, C_Y \le \infty$. Then

$$\begin{aligned} \limsup_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}(\|X+Y\| \le \varepsilon) &\le -\max(C_X, C_Y) \\ \liminf_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}(\|X+Y\| \le \varepsilon) &\ge -\left(C_X^{1/(1+\gamma)} + C_Y^{1/(1+\gamma)}\right)^{1+\gamma}. \end{aligned}$$

Open: What is the precise constant?

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with $0 < \gamma < \infty$ and $0 \le C_X, C_Y \le \infty$. Then

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Open: What is the precise constant?

Thm: If two joint Gaussian random vectors X and Y, *not* necessarily independent, satisfy

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}(\|X\| \le \varepsilon) = -C_X, \quad \lim_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}(\|Y\| \le \varepsilon) = 0$$

with $0 < \gamma < \infty$, $0 < C_X < \infty$. Then

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}(\|X+Y\| \le \varepsilon) = -C_X.$$

Proof of the Lower Bound

For any $0 < \delta < 1$, $0 < \lambda < 1$,

$$\begin{aligned} & \mathbb{P}\left(\|X+Y\| \le \varepsilon\right) \\ \ge\ & \mathbb{P}\left(\|X\| \le (1-\delta)\varepsilon,\ \|Y\| \le \delta\varepsilon\right) \\ \ge\ & \mathbb{P}\left(\|X\| \le \lambda(1-\delta)\varepsilon\right) \cdot \mathbb{P}\left(\|Y\| \le (1-\lambda^2)^{1/2}\delta\varepsilon\right). \end{aligned}$$

Thus

$$\liminf_{\varepsilon\to 0} \varepsilon^{\gamma} \log \mathbb{P}\left(\|X+Y\| \le \varepsilon\right) \ge -(\lambda(1-\delta))^{-\gamma} C_X$$

and the lower bound follows by taking $\delta \to 0$ and $\lambda \to 1$.

•How should we use the inequality for the upper bound?

Proof of the Upper Bound

For the upper bound, we have

$$
\begin{aligned}
& \mathbb{P}\left(\|X\| \leq \frac{\varepsilon}{(1-\delta)\lambda}\right) \\
\geq\ & \mathbb{P}\left(\|X+Y\| \leq \frac{\varepsilon}{\lambda}, \|Y\| \leq \delta \cdot \frac{\varepsilon}{(1-\delta)\lambda}\right) \\
\geq\ & \mathbb{P}\left(\|X+Y\| \leq \varepsilon\right) \cdot \mathbb{P}\left(\|Y\| \leq (1-\lambda^2)^{1/2}\delta \frac{\varepsilon}{(1-\delta)\lambda}\right).
\end{aligned}
$$

Thus

$$
\limsup_{\varepsilon \to 0} \varepsilon^{\gamma} \log \mathbb{P}\left(\|X+Y\| \leq \varepsilon\right) \leq -(\lambda(1-\delta))^{\gamma} C_X
$$

and the upper bound follows by taking $\delta \to 0$ and $\lambda \to 1$.

Applications

As a direct consequence, we have the following for any Gaussian "bridge".

Cor: Let $\{X(t), 0 \le t \le 1\}$ be a $\mathbb{R}^d$-valued, $d \ge 1$, continuous Gaussian random variable. Assume for some norm $\|\cdot\|$ on $C([0,1], \mathbb{R}^d)$ that

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma} \log \mathbb{P}\left(\|X(t)\| \le \varepsilon\right) = -C_X$$

with $0 < \gamma < \infty$ and $0 < C_X < \infty$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma} \log \mathbb{P}\left(\|X(t) - tX(1)\| \le \varepsilon\right) = -C_X.$$

•Brownian sheet and tied down brownian sheet has the same small ball rate under $L_p([0,1]^d)$-norm, $0 \le p \le \infty$.

Our next application extends the small ball results for Brownian motion under weighted sup-norms over the finite interval to those over the infinite interval.

•Let $W(t)$, $t \geq 0$, be the standard Brownian motion. If $f : (0, T] \mapsto (0, \infty)$ satisfies either of the conditions (H1): $\inf_{0<t\leq T} f(t) > 0$ or (H2): $f(t)$ is nondecreasing in a neighborhood of 0. Then,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t\leq T} \frac{|W(t)|}{f(t)} \leq \varepsilon \right) = -\frac{\pi^2}{8} \int_0^T f^{-2}(t)dt.$$

This result was proved by Mogulskii (1974) under essentially condition (H1) and by Berthet and Shi (1998) under condition (H2). The critical case, when $\int_0^T f^{-2}(t)dt = \infty$, and connections with Gaussian Markov processes were treated in Li (1998). Here we extend the result to sup over the whole positive half line.

Thm: Let $g : (0, \infty) \mapsto (0, \infty]$ satisfies the conditions:
(i). $\inf_{0<t<\infty} g(t) > 0$ or $g(t)$ is nondecreasing in a neighborhood of 0.
(ii). $\inf_{0<t<\infty} t^{-1}g(t) > 0$ or $t^{-1}g(t)$ is nonincreasing for t sufficiently large;
Then,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left(\sup_{0<t<\infty} \frac{|W(t)|}{g(t)} \leq \varepsilon \right) = -\frac{\pi^2}{8} \int_0^\infty g^{-2}(t) dt.$$

Here we use the convention $1/\infty = 0$ and hence we can recover the finite interval result by taking $g(t) = f(t)$ for $t \leq T$ and $g(t) = \infty$ for $t > T$.
•Related result for symmetric stable processes will be discussed in lecture 6, together with BM under weighted L_p-norm, $1 \leq p < \infty$.

Without loss of generality, we assume $\int_0^\infty g^{-2}(t)dt$ exists and is finite. The upper estimate follows easily from finite interval case by observing

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t<\infty} \frac{|W(t)|}{g(t)} \le \varepsilon\right) \\
\le\ &\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t\le T} \frac{|W(t)|}{g(t)} \le \varepsilon\right) \\
=\ &-\frac{\pi^2}{8}\int_0^T g^{-2}(t)dt
\end{aligned}$$

for any $T > 0$. Taking $T \to \infty$ gives the desired upper bound. For the lower bound, we have by the weaker correlation inequality with any $0 < \lambda < 1$ and $T > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0<t<\infty} \frac{|W(t)|}{g(t)} \le \varepsilon\right) = \mathbb{P}\left(\sup_{0<t\le T} \frac{|W(t)|}{g(t)} \le \varepsilon\ ,\ \sup_{T\le t<\infty} \frac{|W(t)|}{g(t)} \le\right. \\
\ge\ &\mathbb{P}\left(\sup_{0<t\le T} \frac{|W(t)|}{g(t)} \le \lambda\varepsilon\right) \cdot \mathbb{P}\left(\sup_{T\le t<\infty} \frac{|W(t)|}{g(t)} \le (1-\lambda^2)^{1/2}\varepsilon\right).
\end{aligned}$$

For the second term in the equation above, we have by using the time inversion representation $\{W(t), t>0\} = \{tW(1/t), t>0\}$ in law

$$\mathbb{P}\left(\sup_{T\le t<\infty}\frac{|W(t)|}{g(t)} \le (1-\lambda^2)^{1/2}\varepsilon\right) = \mathbb{P}\left(\sup_{0<t\le 1/T}\frac{|W(t)|}{tg(1/t)} \le (1-\lambda^2)^{1/2}\right.$$

Combining things together, we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\left(\sup_{0<t<\infty}\frac{|W(t)|}{g(t)}\le\varepsilon\right)\\
\ge\;& \liminf_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}\left(\sup_{0<t\le T}\frac{|W(t)|}{g(t)}\le\lambda\varepsilon\right) + \liminf_{\varepsilon\to 0}\varepsilon^2\mathbb{P}\left(\sup_{0<t\le 1/T}\cdots\right)\\
=\;& -\lambda^{-2}\frac{\pi^2}{8}\int_0^T\frac{dt}{g^2(t)} - (1-\lambda^2)^{-1}\frac{\pi^2}{8}\int_0^{1/T}\frac{dt}{t^2g^2(1/t)}\\
=\;& -\lambda^{-2}\frac{\pi^2}{8}\int_0^T\frac{dt}{g^2(t)} - (1-\lambda^2)^{-1}\frac{\pi^2}{8}\int_T^{\infty}\frac{dt}{g^2(t)}.
\end{aligned}$$

Taking $T\to\infty$ first and then $\lambda\to 1$, we obtain the desired lower estimate and thus finish the whole proof.

Inequality in Chen and Li (2003)

Let X and Y be any two centered independent Gaussian random vectors in a separable Banach space B with norm $\|\cdot\|$. We use $|\cdot|_{\mu(X)}$ to denote the inner product norm induced on the associated reproducing Hilbert space H_μ by $\mu = \mathcal{L}(X)$. Then for any $\lambda > 0$ and $\epsilon > 0$,

$$\mathbb{P}(\|Y\| \le \epsilon) \ge \mathbb{P}(\|X\| \le \lambda\epsilon) \cdot \mathbb{E}\exp\{-2^{-1}\lambda^2|Y|^2_{\mu(X)}\}.$$

In particular, for any $\lambda > 0$, $\varepsilon > 0$ and $\delta > 0$,

$$\mathbb{P}(\|Y\| \le \varepsilon) \cdot \exp\{-\lambda^2\delta^2/2\} \ge \mathbb{P}(\|X\| \le \lambda\varepsilon)\,\mathbb{P}\left(|Y|_{\mu(X)} \le \delta\right).$$

$\bullet$This inequality provides a powerful way to estimate lower bound for the class of processes $Y_t = G(C_t), t \in T$ where $G(\cdot)$ is a Gaussian process and C_t is an independent 'clock'. More details in lecture 7.