Ten Lectures on Small Value Probabilities and Applications

L10: Lower Tail Probabilities

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There are only a handful of known examples (specific Gaussian processes) for the one-sided exit asymptotics and it is intellectual challenging to workout more examples in order to find a theory. Here we show the only general result known that can provide sharp estimates at the log level.

The Lower Tail Probability

Let $X = (X_t)_{t\in T}$ be a real valued Gaussian process indexed by T. The lower tail probability studies

$$\mathbb{P}\left(\sup_{t\in T}(X_t - X_{t_0}) \leq \varepsilon\right) \text{ as } \varepsilon \to 0$$

with $t_0 \in T$ fixed.

• Known cases: Brownian motion(BM), Brownian bridge, OU process, integrated BM, fractional BM, and a few more.

• The rate for the integrated fractional Brownian motion is related to the singularity of Burger's equation, See Sinai (1992), Molchan (1999, 2001, 2004, 2006).

• The rate for the m-th integrated Brownian motion is related to the positivity exponent of random polynomials.

•For d-dimensional Brownian sheet $W(t)$, $t \in \mathbb{R}^d$,

$$\log \mathbb{P}\left(\sup_{t\in[0,1]^d} W(t) \leq \varepsilon\right) \approx -\log^d \frac{1}{\varepsilon}.$$

•Many open problems remain and new techniques are needed.

Lower tails for fractional BM

A fractional Brownian motion (FBM) B^H is a centered Gaussian process with covariance

$$\mathbb{E}\, B_t^H B_s^H = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \qquad t, s \in \mathbb{R},$$

where $0 < H < 1$ is the Hurst parameter. For $H = 1/2$, this is a Brownian motion.

•Molchan (1999), Aurzada (2011): For fractional Brownian motion we have, for some $c > 0$,

$$T^{-(1-H)}(\log T)^{-c} \le \mathbb{P}\left(\sup_{0\le t\le T} B_t^H \le 1\right) \le T^{-(1-H)}(\log T)^{c},$$

•**Open:** Remove c.

•**Conj:** $\mathbb{P}\left(\sup_{0\le t\le 1} \int_0^t B_s^H ds \le \varepsilon\right) = \varepsilon^{\frac{H(1-H)}{1+H}+o(1)}$, see Molchan and Khokhlov (2003+) for numerical evidence.

•Aurzada and Simon (2012+), A survey paper on lower tails.

Integrated processes: Simply integrated r.w.

•Aurzada and Dereich (2011+): For X a Levy process or RW with $\exists\beta > 0$: $\mathbb{E}\, e^{\beta|X_1|} < \infty$ and $\mathbb{E}\, X_1 = 0$

$$\mathbb{P}\left(\sup_{0\le n\le T}\sum_{i=1}^{n} X_i \le 1\right) \approx \mathbb{P}\left(\sup_{0\le t\le T}\int_0^t X_s ds \le 1\right) = T^{-1/4} e^{O(\log\log T)}$$

•Dembo and Gao (2011+): For X a RW with $\exists\beta > 0$: $\mathbb{E}\, e^{\beta X_1^-} < \infty$, $\mathbb{E}\, X_1 = 0$, (+ some regularity cond. for X_1^-),

$$\mathbb{P}\left(\sup_{0\le n\le T}\sum_{i=1}^{n} X_i \le 1\right) \approx \sqrt{\frac{\mathbb{E}\,|X_T|}{T\,\mathbb{E}\,|X_1|}} \approx \begin{cases} T^{-1/4} & \text{if } \mathbb{E}\,(X_1^+)^2 < \infty \\ T^{-(1-1/\alpha)/2} & \text{if } X_1^+ \text{ in DoA}(\alpha) \end{cases}$$

•Vysotsky (2011+): For a couple of special cases (all require $\exists\beta > 0$: $\mathbb{E}\, e^{\beta X_1^-} < \infty$, $\mathbb{E}\, X_1 = 0$),

$$\mathbb{P}\left(\sup_{0\le n\le T}\sum_{i=1}^{n} X_i \le 1\right) \sim c T^{-(1-1/\alpha)/2},$$

if X_1^+ in DoA(α), $1 < \alpha \le 2$.

•**Conj:** The rate $1/4$ holds under $\mathbb{E}\, X_1 = 0$ and $\mathbb{E}\, X_1^2 < \infty$.

Slepian process

Here is an old problem of the first passage time for the so-called Slepian process. Let $S(t)$, $t \geq 0$, be the Slepian process, which is the Gaussian process with mean zero and covariance $\mathbb{E}\, S(t)S(s) = (1 - |t-s|)1_{\{|t-s|\leq 1\}}$. It is easy to see that $S(t)$ can be represented in terms of the standard Wiener process $W(t)$ by

$$S(t) = W(t) - W(t+1), \qquad t \geq 0.$$

The first passage probability

$$Q_a(T) = \mathbb{P}\left(\sup_{0\leq t\leq T} S(t) \leq a \right)$$

was studied by many authors. In particular, Slepian (1961) found a simple expression when $T \leq 1$ and Shepp (1971) gave an explicit but very hard to evaluate formula in terms of a T-fold integral for an integer T and a $(2[T]+2)$-fold integral for a non-integer T.

•**Open:** Find the limit (exists by sub-additity)

$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}\left(\sup_{0\leq t\leq n} S(t) \leq a \right)$$

Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ be i.i.d.$N(0,1)$ random variables. Define the random polynomial

$$f_n(x) := \sum_{i=0}^{n} a_i x^i \ .$$

Let N_n denote the number of real zeros of $f_n(x)$.
Dembo, Poonen, Shao and Zeitouni (2002): For n even,

$$\mathbb{P}(N_n = 0) = \mathbb{P}(f_n(x) > 0, \, \forall x \in \mathbb{R}) = n^{-b+o(1)}$$

where

$$b = -4 \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}\Big(\sup_{0 \le s \le t} Y(s) \le 0 \Big)$$

and $\{Y(t), \ t \ge 0\}$ is a centered stationary Gaussian process with

$$\mathbb{E}\, Y(t) Y(s) = \frac{2e^{-(t-s)/2}}{1 + e^{-(t-s)}}$$

Moreover,

$$0.4 < b \le 2.$$

Their numerical simulations for degree $n \le 2^{10}$ suggest $b \approx 0.76 \pm 0.03$.

Let $W(t)$, $t \geq 0$, be the standard Brownian motion starting at 0. Denote by $W_0(t) = W(t)$ and

$$W_m(t) = \int_0^t W_{m-1}(s)ds, \quad t \geq 0, \quad m \geq 1.$$

the m times integrated Brownian motion for positive integer m. Using integration by parts,

$$W_m(t) = \frac{1}{m!}\int_0^t (t-s)^m dW(s), \quad m \geq 0.$$

The Gaussian process $W_m(t)$ has been studied from various points of view in Shepp (1966), Wahba (1978), Lachal (1997) and Chen and Li (2002). The $\mathbb{R}^{m+1}$ valued process $(W_0(t), W_1(t), \cdots, W_m(t))$ is Markov with degenerated generator

$$\mathcal{L} = \frac{\partial^2}{\partial x_0^2} + \sum_{k=1}^{m} x_{k-1}\frac{\partial}{\partial x_k}.$$

When $m = 1$, the process

$$(W_0(t), W_1(t)) = (W(t), \int_0^t W(s)ds)$$

Li and Shao (2007+), Aurzada, Li and Shao (2012+): There exist constants $r_m >$ and $r > 0$ such that

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le s\le \log t} Y(s) \le 0\right) &\approx t^{-r+o(1)}, \\
\mathbb{P}\left(\sup_{0\le s\le t} X(s) \le 1\right) &\approx t^{-r+o(1)}, \\
\mathbb{P}\left(\sup_{0\le t\le 1} X(t) \le \varepsilon\right) &\approx \varepsilon^{2r+o(1)}, \\
\mathbb{P}\left(\sup_{0\le t\le 1} W_m(t) \le \varepsilon\right) &\approx \varepsilon^{r_m+o(1)}, \\
\mathbb{P}\left(\sup_{0\le s\le t} W_m(s) \le 1\right) &\approx t^{-r_m(2m+1)/2+o(1)}
\end{aligned}$$

and $r_m(2m+1)/2$ decrease to r as $m \to \infty$, where $X(t)$ is a centered Gaussian process with $\mathbb{E}\, X(t)X(s) = \frac{2st}{t+s}$. In particular, $b = 4r \le 1$ since $r_1 = 1/6$ from McKean (1963), Sinai (1992). Note that $r_0 = 1$ and the scaling $W_m(ct) = c^{(2m+1)/2} W_m(t)$. The problems of finding r and r_m, $m \ge 2$ are open.

Uniformly Dudley type entropy

Let $X = (X_t)_{t \in T}$ be a real valued Gaussian random process indexed by T with mean zero. Define the L^2-metric

$$d(s,t) = (\mathbb{E}\,|X_s - X_t|^2)^{1/2}, \quad s,t \in T.$$

For every $\varepsilon > 0$ and a subset A of T, let $N(A, \varepsilon)$ denote the minimal number of open balls of radius ε for the metric d that are necessary to cover A.

For $t \in T$ and $h > 0$, let $B(t,h) = \{s \in T : d(t,s) \leq h\}$, and define a locally and uniformly Dudley type entropy (LUDE) integral

$$Q = \sup_{h>0} \sup_{t \in T} \int_0^\infty (\log N(B(t,h), \varepsilon h))^{1/2} d\varepsilon$$

•We follow Li and Shao (2004) for the rest of this lecture.

An Lower Bound

Assume that $Q < \infty$ and $t_0 \in T$. For $\theta = 1000(1+Q)$, define

$$\begin{aligned} A_{-1} &= \{t \in T : d(t, t_0) \le \theta^{-1} x\}, \\ A_k &= \{t \in T : \theta^{k-1} x < d(t, t_0) \le \theta^k x\}, \end{aligned}$$

where $0 \le k \le L$, $L = 1 + [\log_\theta(D/x)]$ and $D = \sup_{t \in T} d(t, t_0)$. Let $N_k(x) := N(A_k,\ \theta^{k-2} x)$, $k = 0, 1, ..., L$, and

$$N(x) = 1 + \sum_{0 \le k \le L} N_k(x).$$

Thm: Assume that $Q < \infty$ and

$$\mathbb{E}\left((X_s - X_{t_0})(X_t - X_{t_0})\right) \ge 0 \quad \text{for} \quad s, t \in T$$

Then we have

$$\mathbb{P}\left(\sup_{t \in T} X_t - X_{t_0} \le x\right) \ge e^{-N(x)}$$

Pf: We start with a basic application of Slepian's lemma. Assume $\mathbb{E}\, X_t X_s \geq 0$ and $T = A_1 \cup A_2$. Then

$$\mathbb{P}(\sup_{t\in T} X_t \leq x) \geq \mathbb{P}(\sup_{t\in A_1} X_t \leq x) \cdot \mathbb{P}(\sup_{t\in A_2} X_t \leq x).$$

To see this clearly, we define the comparison process

$$Y(t) = \begin{cases} X(t), & t \in A_1 \\ X^*(t), & t \in A_1^c A_2. \end{cases}$$

where $X^*(t)$ is ind. of $X(t)$ and has the same distribution as $X(t)$ as a process. Then the condition of Slepian's lemma is satisfied. Thus

$$\mathbb{P}(\sup_{t\in T} X_t \leq x) \geq \mathbb{P}(\sup_{t\in A_1} X_t \leq x) \cdot \mathbb{P}(\sup_{t\in A_1^c A_2} X_t^* \leq x)$$

and

$$\mathbb{P}(\sup_{t\in A_1^c A_2} X_t^* \leq x) = \mathbb{P}(\sup_{t\in A_1^c A_2} X_t \leq x) \geq \mathbb{P}(\sup_{t\in A_2} X_t \leq x)$$

Without loss of generality, assume that $X_{t_0} = 0$. Let $A_{k,j}, j = 1, ..., N_k(x)$ be the open balls of radius $\theta^{k-2}x$ for the metric d that cover A_k, $k = 0, 1, ..., L$. Then, by the positive correlation assumption and the Slepian's lemma

$$\mathbb{P}\left(\sup_{t\in T}(X_t - X_{t_0}) \leq x\right) \geq \mathbb{P}\left(\sup_{t\in A_{-1}} X_t \leq x\right) \prod_{k=0}^{L} \prod_{j=1}^{N_k(x)} \mathbb{P}\left(\sup_{t\in A_{k,j}} X_t \leq x\right)$$

By Dudley (1967), we have

$$\begin{aligned}
\mathbb{E} \sup_{t\in A_{-1}} X(t) &\leq 42 \int_0^{\theta^{-1}x} (\log N(A_{-1}, \varepsilon))^{1/2} d\varepsilon \\
&\leq 42 \int_0^{\theta^{-1}x} (\log N(B(t_0, \theta^{-1}x), \varepsilon))^{1/2} d\varepsilon \\
&= 42\theta^{-1}x \int_0^{1} (\log N(B(t_0, \theta^{-1}x), \varepsilon\theta^{-1}x))^{1/2} d\varepsilon \\
&\leq 42Q\theta^{-1}x \leq x/2.
\end{aligned}$$

Hence $\mathbb{P}\left(\sup_{t\in A_{-1}} X_t \leq x\right) = 1 - \mathbb{P}\left(\sup_{t\in A_{-1}} X_t > x\right) \geq 1/2.$

It suffices to show that

$$\mathbb{P}\left(\sup_{t\in A_{k,j}} X_t \le 0\right) \ge e^{-1}$$

for every $1 \le j \le N_k(x), 0 \le k \le L$. Let $s_{k,j}$ be the center of $A_{k,j}$. Then $d(s_{k,j}, t_0) \ge \theta^{k-1}x$.
Observe that

$$\begin{aligned} & \mathbb{P}\left(\sup_{t\in A_{k,j}} X_t \le 0\right) \\ \ge\ & \mathbb{P}\left(X_{s_{k,j}} \le -\theta^{k-1}x/4\right) - \mathbb{P}\left(\sup_{t\in A_{k,j}} (X_t - X_{s_{k,j}}) > \theta^{k-1}x/4\right) \end{aligned}$$

and

$$\mathbb{P}\left(X_{s_{k,j}} \le -\theta^{k-1}x/4\right) \ge \mathbb{P}\left(Z \ge 1/4\right) \ge e^{-1} + 10^{-2}.$$

where Z is $N(0,1)$.

By the definition of $A_{k,j}$, we have

$$\sup_{t \in A_{k,j}} d(t, s_{k,j}) \leq \theta^{k-2} x$$

and similar to the A_{-1} case,

$$\mathbb{E} \sup_{t \in A_{k,j}} (X(t) - X_{s_{k,j}}) \leq 42 Q \theta^{k-2} x.$$

Hence, it follows from the deviation estimate for Gaussian process,

$$\begin{aligned}
& \mathbb{P}\left(\sup_{t \in A_{k,j}} (X_t - X_{s_{k,j}}) > \theta^{k-1} x/4 \right) \\
\leq \ & \mathbb{P}\Bigg(\sup_{t \in A_{k,j}} (X_t - X_{s_{k,j}}) > \mathbb{E} \sup_{t \in A_{k,j}} (X_t - X_{s_{k,j}}) + \theta^{k-1} x/4 - 42 Q \theta^{k-2} x \\
\leq \ & \mathbb{P}\left(\sup_{t \in A_{k,j}} (X_t - X_{s_{k,j}}) > \mathbb{E} \sup_{t \in A_{k,j}} (X_t - X_{s_{k,j}}) + \theta^{k-1} x/5 \right) \\
\leq \ & 2 \exp(-(\theta/5)^2/2) \leq 10^{-2}
\end{aligned}$$

since $\theta = 1000(1+Q) \geq 1000$.

An Upper Bound

For $x > 0$, let $s_i \in T$, $i = 1, ..., M$ be a sequence such that for every i

$$\sum_{j=1}^{M} |\mathrm{Corr}(X_{s_i} - X_{t_0}, X_{s_j} - X_{t_0})| \leq 5/4$$

and

$$d(s_i, t_0) = (\mathbb{E}\,|X_{s_i} - X_{t_0}|^2)^{1/2} \geq x/2.$$

Then

$$\mathbb{P}\left(\sup_{t \in T} X_t - X_{t_0} \leq x\right) \leq e^{-M/10}.$$

A Simple Comparison

Let $\mathbf{X} = (X_1, ..., X_n)'$ be distributed according to $N(\mathbf{0}, \mathbf{\Sigma_1})$, and $\mathbf{Y} = (Y_1, ..., Y_n)'$ according to $N(\mathbf{0}, \mathbf{\Sigma_2})$. If $\mathbf{\Sigma_2} - \mathbf{\Sigma_1}$ is positive semidefinite, then for all $C \subset \mathbb{R}^n$,

$$\mathbb{P}\left(\mathbf{Y} \in C\right) \geq (|\mathbf{\Sigma_1}|/|\mathbf{\Sigma_2}|)^{\mathbf{1/2}} \mathbb{P}(\mathbf{X} \in \mathbf{C}).$$

A Simple Comparison

Let $\mathbf{X} = (X_1, ..., X_n)'$ be distributed according to $N(\mathbf{0}, \mathbf{\Sigma_1})$, and $\mathbf{Y} = (Y_1, ..., Y_n)'$ according to $N(\mathbf{0}, \mathbf{\Sigma_2})$. If $\mathbf{\Sigma_2} - \mathbf{\Sigma_1}$ is positive semidefinite, then for all $C \subset \mathbb{R}^n$,

$$\mathbb{P}\left(\mathbf{Y} \in C\right) \geq (|\mathbf{\Sigma_1}|/|\mathbf{\Sigma_2}|)^{\mathbf{1/2}} \mathbb{P}(\mathbf{X} \in \mathbf{C}).$$

Pf: Let $f_{\mathbf{X}}$ and $f_{\mathbf{Y}}$ be the joint density functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Since $\mathbf{\Sigma_2} - \mathbf{\Sigma_1}$ is positive semidefinite, $\mathbf{\Sigma_1^{-1}} - \mathbf{\Sigma_2^{-1}}$ is positive semidefinite too, see Bellman (1970), page 59. Hence

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{x}) &= \frac{1}{(2\pi)^{n/2}|\mathbf{\Sigma_2}|^{\mathbf{1/2}}} \exp\Big(-\frac{1}{2}\mathbf{x}'\mathbf{\Sigma_2^{-1}}\mathbf{x}\Big) \\ &\geq \frac{1}{(2\pi)^{n/2}|\mathbf{\Sigma_2}|^{\mathbf{1/2}}} \exp\Big(-\frac{1}{2}\mathbf{x}'\mathbf{\Sigma_1^{-1}}\mathbf{x}\Big) \\ &= (|\mathbf{\Sigma_1}|/|\mathbf{\Sigma_2}|)^{\mathbf{1/2}} \mathbf{f_X}(\mathbf{x}). \end{aligned}$$

Pf of Thm:. Without loss of generality, assume again that $X_{t_0} = 0$. Let $Z_i, 1 \leq i \leq M$ be i.i.d. standard normal random variables, $\mathbf{\Sigma_1}$ be the covariance matrix of

$$\{X_{s_i}/(\mathbb{E}\,|X_{s_i}|^2)^{1/2}, 1 \leq i \leq M\}$$

and $\mathbf{\Sigma_2}$ be the covariance matrix of $\{3Z_i/2, 1 \leq i \leq M\}$. By the assumption

$$\sum_{j=1}^{M} |\mathrm{Corr}(X_{s_i}, X_{s_j})| \leq 5/4,$$

$\mathbf{\Sigma_1}$ is a dominant principal diagonal matrix. Moreover, by Price (1951)

$$\det(\mathbf{\Sigma_1}) \geq (1 - 1/4)^M$$

It follows from the assumption again that $\mathbf{\Sigma_2} - \mathbf{\Sigma_1}$ is also a dominant principal diagonal matrix and hence is positive semidefinite.

Thus for all $G \subset \mathbb{R}^M$,

$$\begin{aligned} & \mathbb{P}\left((X_{s_i}/(\mathbb{E}\,X_{s_i}^2)^{1/2}, i \leq M) \in G\right) \\ \leq & \left(\det(\mathbf{\Sigma_2})/\det(\mathbf{\Sigma_1})\right)^{1/2} \mathbb{P}\left((3Z_i/2, i \leq M) \in G\right) \\ \leq & \; 2^{M/2} \mathbb{P}\left((3Z_i/2, i \leq M) \in G\right). \end{aligned}$$

In particular, we have

$$\begin{aligned} \mathbb{P}\left(\max_{i \leq M} X_{s_i} \leq x\right) &= \mathbb{P}\left(\bigcap_{i \leq M} \{X_{s_i}/(\mathbb{E}\,X_{s_i}^2)^{1/2} \leq x/(\mathbb{E}\,X_{s_i}^2)^{1/2}\right) \\ &\leq \mathbb{P}\left(\bigcap_{i \leq M} \{X_{s_i}/(\mathbb{E}\,X_{s_i}^2)^{1/2} \leq 1/2\right) \\ &\leq 2^{M/2} \mathbb{P}\left(\max_{i \leq M} 3Z_i/2 \leq 1/2\right) \\ &= \left(2^{1/2} \mathbb{P}\left(Z \leq 1/3\right)\right)^M \\ &\leq e^{-M/10}. \end{aligned}$$

Cor: Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and stationary increments, that is

$$\forall\ t, s \in [0,1]^d, \quad \mathbb{E}\,(X_t - X_s)^2 = \sigma^2(|t-s|)$$

where $|\cdot|$ is Euclidean norm on $\mathbb{R}^d$. If there are $0 < \alpha \le \beta < 1$ such that $\sigma(h)/h^\alpha$ is non-decreasing and $\sigma(h)/h^\beta$ non-increasing. Then there exist $0 < c_1 \le c_2 < \infty$ depending only on α, β and d such that for $0 < x < 1/2$

$$-c_2 \log \frac{1}{x} \le \log \mathbb{P}\left(\sup_{t \in [0,1]^d} X(t) \le \sigma(x) \right) \le c_1 \log \frac{1}{x}.$$

In particular we have for the fractional Levy's Brownian motion $L_\alpha(t)$ of order α, i.e. $L_\alpha(0) = 0$ and $\mathbb{E}\,(L_\alpha(t) - L_\alpha(s))^2 = |t-s|^\alpha$, $0 < \alpha < 2$,

$$-c_2 \log \frac{1}{x} \le \log \mathbb{P}\left(\sup_{t \in [0,1]^d} L_\alpha(t) \le \sigma(x) \right) \le c_1 \log \frac{1}{x}.$$

Cor: Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and

$$\mathbb{E}\left(X_t X_s\right) = \prod_{i=1}^{d} \frac{1}{2}(\sigma^2(t_i) + \sigma^2(s_i) - \sigma^2(|t_i - s_i|))$$

for $t = (t_1, ..., t_d)$ and $s = (s_1, ..., s_d)$, where σ is a nondecreasing function.
If there are $0 < \alpha \le \beta < 1$ such that $\sigma(h)/h^\alpha$ is non-decreasing and $\sigma(h)/h^\beta$ non-increasing. Then there exist $0 < c_3 \le c_4 < \infty$ depending only on α, β and d such that for $0 < x < 1/2$

$$-c_4 \log^d \frac{1}{x} \le \log \mathbb{P}\left(\sup_{t\in[0,1]^d} X(t) \le \sigma^d(x)\right) \le -c_3 \log^d \frac{1}{x}.$$

In particular, for d-dimensional Brownian sheet $W(t)$, $t \in \mathbb{R}^d$,

$$\log \mathbb{P}\left(\sup_{t\in[0,1]^d} X(t) \le x\right) \approx -\log^d \frac{1}{x}.$$

Proof of the upper bound for d-dimensional Brownian sheet
Let $\theta > 1$, $L = [\log_\theta(1/x)]$ and

$$s_{\mathbf{k}} = \theta^{\mathbf{k}} x^{1/d}, \quad \mathbf{k} = (k_1, ..., k_d), 1 \le k_i \le L$$

so that $d(s_{\mathbf{k}}, \mathbf{0}) = \theta^{k_1+\cdots+k_d} x \ge x/2$. Note that

$$\begin{aligned} |\mathrm{Corr}(X_{s_{\mathbf{k}}}, X_{s_{\mathbf{j}}})| &= \prod_{i=1}^{d} \min(\theta^{(k_i-j_i)/2}, \theta^{(j_i-k_i)/2}) \\ &= \theta^{-\sum_{i=1}^{d} |k_i-j_i|/2}. \end{aligned}$$

Therefore for any given $\mathbf{k}$

$$\begin{aligned} \sum_{1\le \mathbf{j} \le L} |\mathrm{Corr}(X_{\mathbf{k}}, X_{\mathbf{j}})| &\le \sum_{1\le \mathbf{j} \le L} \theta^{-\sum_{i=1}^{d} |k_i-j_i|/2} \\ &\le 1 + \frac{2^d}{\theta^{1/2}(1-\theta^{-1/2})^d} \le 5/4 \end{aligned}$$

for θ sufficiently large.

Why lower tail is harder?

Note that we can write

$$\|X\| = \sup_{f \in D} f(X)$$

so the lower tail formulation is more general than the small ball problem.

Open: Are there any connections with properties of the generating compact operator?

Probability of all real zeros for random polynomial with exponential ensemble

Thm: (Li (2012)). The probability that a random polynomial of degree n with i.i.d exponentially distributed coefficients has all real zeros is

$$\mathbb{P}(\text{All zeros are real}) = \mathbb{E} \prod_{1 \le j < k \le n} |U_j - U_k| = \left(\prod_{k=1}^{n-1} \binom{2k+1}{k} \right)^{-1}$$

where U_i are i.i.d uniform on the interval $[0, 1]$.
•In particular, we have

$$p_1^e = 1, \quad p_2^e = \frac{1}{3}, \quad p_3^e = \frac{1}{30}, \quad p_4^e = \frac{1}{1050} \quad p_5^e = \frac{1}{132300}.$$

•Asymptotically, $\log \mathbb{P}(N_n = n) \sim -\log 2 \cdot n^2$ as $n \to \infty$.
•The second identity is a form of Selberg integral.