Irregularities of Distribution
New Inequalities in all dimensions $d \geq 3$

Dmitriy Bilyk & Michael Lacey & Armen Vagharshakyan

June 7, 2012
For $B_t$ the $d$-dimensional Brownian Sheet, consider

$$- \log P\left( \sup_{t \in [0,1]^d} |B_t| < \epsilon \right) = \phi(\epsilon)$$

- Chung’s Law: For $d = 1$, $\phi(\epsilon) \simeq \epsilon^{-2}$
- Talagrand’s Law: For $d = 2$, $\phi(\epsilon) \simeq \epsilon^{-2}(\log 1/\epsilon)^3$. 
Two Giants: Klaus Roth and Wolfgang Schmidt
Let $\mathcal{P}_N$ be a subset of $[0, 1]^d$ of cardinality $N$.

$$D_N(x) = \#\{\mathcal{P}_N \cap [0, x)\} - N|[0, x)|$$

- A $d$ dimensional box.

**Roth’s Theorem**

*For any choice of $\mathcal{P}_N$ we have*

$$\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}$$
**Theorem**

For any choice of point distribution $P_N$ we have

$$\|D_N\|_p \geq (\log N)^{(d-1)/2}, \quad 1 < p < \infty.$$  

There is however a ‘kink’ at $L^\infty$ in Dimension $d = 2$.

**Schmidt’s Theorem ($d = 2$)**

$$\|D_N\|_{L^\infty([0,1]^2)} \geq \log N$$

A gain of $\sqrt{\log N}$ over the average case bound.
Theorem (Jozef Beck, 1989)

In dimension 3, there holds

\[ \|D_N\|_\infty \gtrsim (\log N)^{(3-1)/2} (\log \log N)^{1/8}. \]
Theorem (Bilyk-L.-Vagharshakyan, 2007)

For $d \geq 3$, there is an $\eta = \eta(d) \geq c / d^2$ for which

$$\|D_N\|_\infty \gtrsim (\log N)^{(d-1)/2 + \eta}.$$ 

A gain of $\eta$ over the Roth bound.
Conjecture: Discrepancy Function in $L^\infty$

For $d \geq 3$,

$$\|D_N\|_\infty \gtrsim \begin{cases} (\log N)^{d-1} \\ (\log N)^{d/2} \end{cases}$$

$d/2$: Supported by analogous conjectures in Stochastic Processes and in Approximation Theory
**Definitions**

**Dyadic Intervals**

\[ \mathcal{D} = \{ [j2^{-n}, (j + 1)2^{-n}) \mid 0 \leq j < 2^n \} , \]

**Product Haar Functions**

For \( R_1, \ldots, R_d \in \mathcal{D}^d \),

\[ h_{R_1 \times \cdots \times R_d}(x_1, \ldots, x_d) = \prod_{j=1}^{d} \left\{ -1_{\text{left}_{j}}(x_j) + 1_{\text{right}_{j}}(x_j) \right\} \]
A product rule holds.

\[ h_R \cdot h_S = -h_{R \cap S} \]
Product Rule Fails in Three Dimensions
Average Case Bound:

\[ \left\| \sum_{|R|=2^{-n}} a_R h_R(x) \right\|_\infty \lesssim n^{(d-1)/2}, \quad a_R \in \{-1, 0, 1\} \]

Conjecture: Small Ball Inequality

For \( d \geq 3 \), and generic choices of coefficients \( a_R \in \{-1, 0, 1\} \),

\[ \left\| \sum_{|R|=2^{-n}} a_R h_R(x) \right\|_\infty \gtrsim n^{d/2} . \]

- \( d = 2 \) is a Theorem of Talagrand.
- Both conjectures are a ‘gain of a square root’ over the average case bounds.
- The \( d/2 \) is sharp.
- For Talagrand Theorem, $d = 2$, every point is in $n + 1$ dyadic rectangles of area $2^{-n}$. Want to find one point where all the haar functions have the same sign.
- For $d = 3$, every point is in $\approx n^2$ dyadic rectangles of volume $2^{-n}$.
- But, the best possible result is to find a single point where the number of Haar functions with a ‘+1’ exceeds the number of ‘−1’s by a factor $n^{3/2}$.
- The ‘surplus’ in percentage terms is only $n^{-1/2}$. 