Irregularities of Distribution
New Inequalities in all dimensions $d \geq 3$

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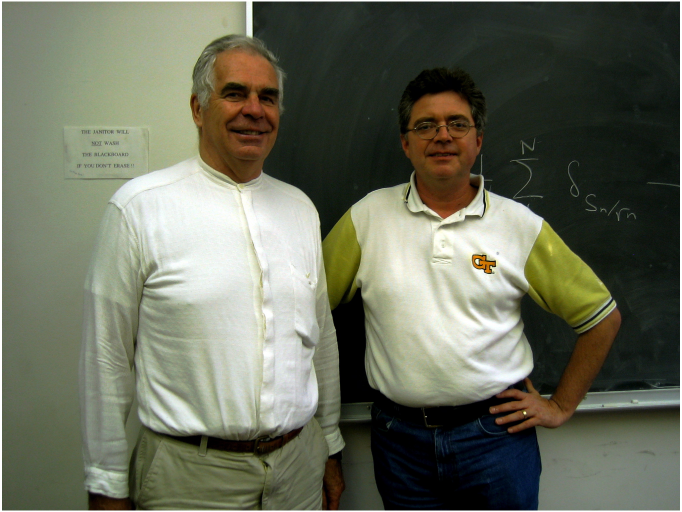

- For B_t the d-dimensional Brownian Sheet, consider

$$-\log P\left(\sup_{t\in[0,1]^d} |B_t| < \epsilon\right) = \phi(\epsilon)$$

- Chung's Law: For $d = 1$, $\phi(\epsilon) \simeq \epsilon^{-2}$
- Talagrand's Law: For $d = 2$, $\phi(\epsilon) \simeq \epsilon^{-2}(\log 1/\epsilon)^3$.

Let $\mathcal{P}_N$ be a subset of $[0,1]^d$ of cardinality N.

$$D_N(x) = \sharp\{\mathcal{P}_N \cap [0,x)\} - N|[0,x)|$$

- A d dimensional box.

Roth's Theorem

For any choice of $\mathcal{P}_N$ we have

$$\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}$$

Theorem

For any choice of point distribution $\mathcal{P}_N$ we have

$$\|D_N\|_p \gtrsim (\log N)^{(d-1)/2}, \qquad 1 < p < \infty .$$

There is however a ‘kink’ at L^∞ in Dimension $d = 2$.

Schmidt’s Theorem ($d = 2!$)

$$\|D_N\|_{L^\infty([0,1]^2)} \gtrsim \boxed{\log N}$$

A gain of $\boxed{\sqrt{\log N}}$ *over the average case bound.*

Theorem (Jozef Beck, 1989)

In dimension 3, there holds

$$\|D_N\|_\infty \gtrsim (\log N)^{(3-1)/2} (\log \log N)^{1/8} .$$

Theorem (Bilyk-L.-Vagharshakyan, 2007)

For $d \geq 3$, there is an $\eta = \eta(d) \geq c/d^2$ for which

$$\|D_N\|_\infty \gtrsim (\log N)^{(d-1)/2+\boxed{\eta}}.$$

A gain of $\boxed{\eta}$ over the Roth bound.

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Conjecture: Discrepancy Function in L^∞

For $d \geq 3$,

$$\|D_N\|_\infty \gtrsim \begin{cases} (\log N)^{d-1} \\ (\log N)^{d/2} \end{cases}$$

$d/2$: Supported by analogous conjectures in Stochastic Processes and in Approximation Theory

Dyadic Intervals

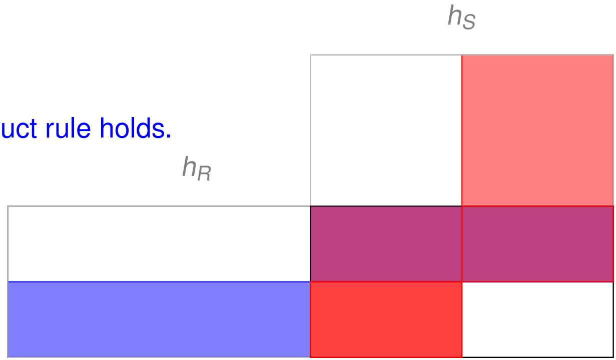
$$\mathcal{D} = \{[j2^{-n}, (j+1)2^{-n}) \mid 0 \leq j < 2^n\},$$

Product Haar Functions

For $R_1, \ldots, R_d \in \mathcal{D}^d$,

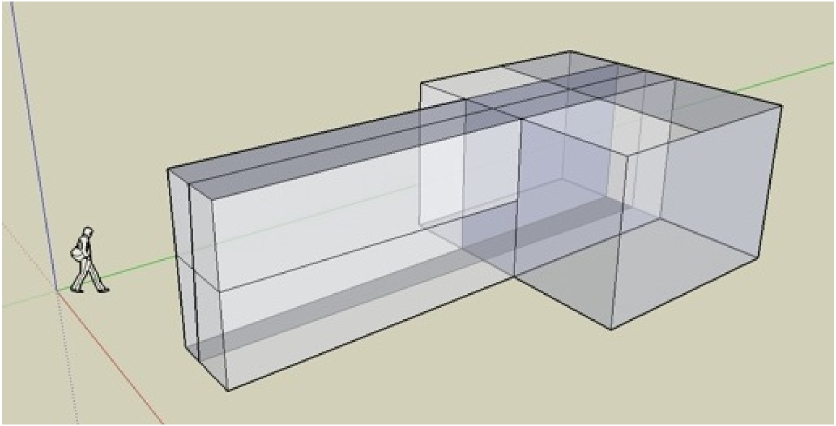
$$h_{R_1 \times \cdots \times R_d}(x_1, \ldots, x_d) = \prod_{j=1}^{d} \{-\mathbf{1}_{I_{j,\text{left}}}(x_j) + \mathbf{1}_{R_{j,\text{right}}}(x_j)\}$$

A product rule holds.



$$h_R \cdot h_S = -h_{R \cap S}$$

Product Rule Fails in Three Dimensions



Average Case Bound:

$$\Big\| \sum_{|R|=2^{-n}} a_R h_R(x) \Big\|_\infty \lesssim n^{(d-1)/2}\,, \qquad a_R \in \{-1, 0, 1\}$$

Conjecture: Small Ball Inequality

For $d \geq 3$, and generic choices of coefficients $a_R \in \{-1, 0, 1\}$,

$$\Big\| \sum_{|R|=2^{-n}} a_R h_R(x) \Big\|_\infty \gtrsim n^{d/2}\,.$$

- $d = 2$ is a Theorem of Talagrand.
- Both conjectures are a 'gain of a square root' over the average case bounds.
- The $d/2$ is sharp.

- For Talagrand Theorem, $d = 2$, every point is in $n + 1$ dyadic rectangles of area 2^{-n}. Want to find one point where all the haar functions have the same sign.
- For $d = 3$, every point is in $\simeq n^2$ dyadic rectangles of volume 2^{-n}.
- But, the best possible result is to find a single point where the number of Haar functions with a '+1' exceeds the number of '-1's by a factor $n^{3/2}$.
- The 'surplus' in percentage terms is only $n^{-1/2}$.