First passage times of Lévy processes over a one-sided moving boundary

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joint works with F. Aurzada, M. Lifshits and M. Savov

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Outline

1. Statement of the problem
2. Brownian motion
   - constant boundaries
   - moving boundaries
3. General Lévy processes
   - constant boundaries
   - moving boundaries
4. strictly stable Lévy processes
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5. Conclusion

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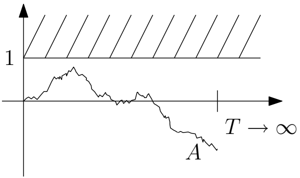
Statement of the problem

Given: $(A_t)_{t\geq 0}$ stochastic process with $A_0 = 0$.
Goal: Find asymptotics of

$$\mathbb{P}\left[\sup_{0\leq t\leq T} A_t \leq 1\right] \approx \ ?\,, \qquad \text{as } T \to \infty.$$



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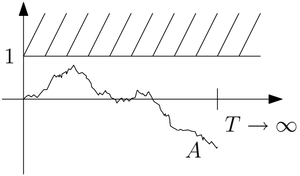


Here, we expect

$$\mathbb{P}\left[\sup_{0\leq t\leq T} A_t \leq 1\right] = T^{-\theta+o(1)}, \qquad \text{as } T \to \infty$$

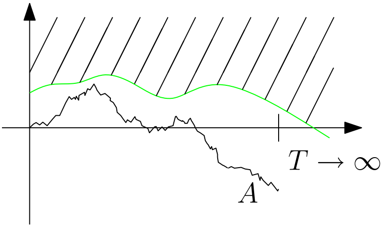
with $\theta > 0$, called survival exponent.

Statement of the problem

The exit problem with a "moving boundary":

$$\mathbb{P}\left[\forall t \in [0,T] \,:\, A_t \leq \mathbf{F}(t)\right] = T^{-\theta+o(1)}$$

Question:
For which F does one get the same survival exponent as for $F \equiv 1$?



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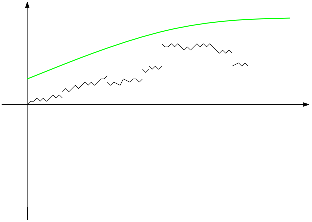

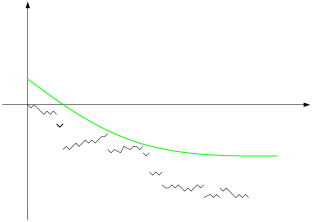

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BM: known results for constant boundaries

Let B be a Brownian motion.

Then,

$$\mathbb{P}\left[\sup_{0\le t\le T} B_t \le 1\right] \sim \sqrt{\frac{2}{\pi}} \cdot T^{-1/2}, \qquad \text{as } T \to \infty,$$

$\rightsquigarrow$ easily proved by the reflection principle.

BM: known results for moving boundaries

Theorem (Uchiyama'80)

If F is continuous and $F(0) > 0$ and such that

$$\int_1^\infty t^{-3/2}|F(t)|\,\mathrm{d}t < \infty$$

then

$$\mathbb{P}\left[\forall 0 \leq t \leq T : B_t \leq F(t)\right] \approx T^{-1/2}.$$

The integral test is in some sense necessary.

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The integral test is in some sense necessary.

- $F(t) = \sqrt{t}$ does not satisfy the integral test, but $F(t) = \sqrt{t}(\log t)^{-\gamma}$, $\gamma > 1$.
- Proof: comparison lemmas for Brownian non-exit probabilities and a "time-discretization" technique.

BM: new approach to the proof

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The integral test is in some sense necessary.

- Novikov (1992) simplified the proof for the increasing boundary using martingale techniques.
- The proof for the decreasing boundary was simplified by Aurzada/K.'12+. The integral test above can be understood and interpreted as a repulsion effect of a Bessel-(3)-process.

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LP: constant boundary

For the rest of the talk, we consider a Lévy process X with common Lévy triplet (b, σ, ν).

- $$\mathbb{P}\left[\sup_{0\leq t\leq T} X_t \leq 1\right], \quad \text{as } T \to \infty$$

  is the subject of study of classical fluctuation theory.

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  is the subject of study of classical fluctuation theory.
- This problem is closely related to the behaviour of
  $$\mathbb{P}[X_t > 0], \qquad \text{as } t \to \infty.$$
  If
  $$\mathbb{P}[X_t > 0] \to \rho, \qquad \text{as } t \to \infty$$
  for some $\rho \in (0,1)$ we say X satisfies Spitzer's condition with parameter $\rho \in (0,1)$.

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- Similarly (and historically earlier), corresponding results for random walks.

LP: constant boundary

Theorem (e.g. Rogozin' 71)

The following assertions are equivalent for each $\rho \in (0,1)$

1. X *satisfies Spitzer's condition with* $\rho \in (0,1)$.
2. $$\mathbb{P}\left[\sup_{0\le t\le T} X_t \le x\right] \sim c(x)T^{-\rho}\ell(T)$$

   with some slowly varying ℓ.

LP: known results for moving boundaries

Theorem (Greenwood/Novikov'86)

Let X be a Lévy process that satisfies Spitzer's condition with $\rho \in (0,1)$. Then

$$\mathbb{P}\left[\forall 0 \leq t \leq T : X_t \leq 1\right] = T^{-\rho+o(1)}$$

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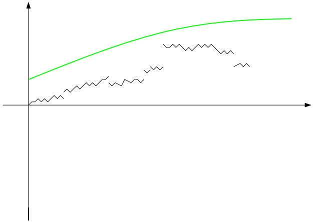
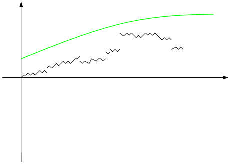
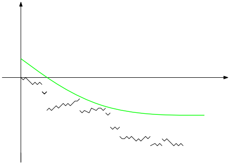
and, for $\gamma < \rho$,

$$\mathbb{P}\left[\forall 0 \le t \le T : X_t \le \mathbf{1} + \mathbf{c}\mathbf{t}^{\gamma}\right] = T^{-\rho+o(1)}.$$



LP: new results for moving boundaries





Theorem (Aurzada/K./Savov'12+)

Let X be a Lévy process and $\gamma < \mathbf{1/2}$. If for some $\rho > 0$

$$\mathbb{P}\left[\forall 0 \le t \le T : X_t \le 1\right] = T^{-\rho+o(1)}$$

and $\nu(-\infty, 0) > 0$ then

$$\mathbb{P}\left[\forall 0 \le t \le T : X_t \le \mathbf{1 - ct^{\gamma}}\right] = T^{-\rho+o(1)}$$

and additionally $\nu(0, \infty) > 0$ then

$$\mathbb{P}\left[\forall 0 \le t \le T : X_t \le \mathbf{1 + ct^{\gamma}}\right] = T^{-\rho+o(1)}.$$

LP: new results for moving boundaries

- Change of measure (Girsanov transform):

$$\mathbb{P}\left[\forall t \leq T : X_t + f(t) \leq 1\right] \geq \mathbb{P}\left[\forall t \leq T : X_t + Z_t \leq 1\right] T^{o(1)} e^{-\frac{1}{2}|f'|^2_{L_2}},$$

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Conjecture for decreasing boundaries

1. Let $\beta_- := \sup\{r \geq 0 : \mathbb{E}\left[(X_1^-)^r\right] < \infty\}$.
2. Let
$$\mathbb{P}\left[\forall 0 \leq t \leq T : X_t \leq 1\right] = T^{-\rho+o(1)},$$
for some $\rho \in (0,1)$, that is X satisfies Spitzer's condition with parameter $\rho \in (0,1)$.

We expect that these assumptions imply

$$\gamma < \max\left\{\frac{1}{2}, \frac{1}{\beta_-}\right\} \iff \mathbb{P}\left[\forall 0 \leq t \leq T : X_t \leq 1 - t^\gamma\right] = T^{-\rho+o(1)}.$$

Conjecture for incresasing boundaries

1. Let $\beta_+ := \sup\{r \geq 0 : \mathbb{E}\left[(X_1^+)^r\right] < \infty\}$.
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$$\gamma < \max\left\{\frac{1}{2}, \frac{1}{\beta_+}\right\} \Longleftrightarrow \mathbb{P}\left[\forall 0 \leq t \leq T : X_t \leq 1 + t^\gamma\right] = T^{-\rho + o(1)}.$$

Recall that $\rho \leq \frac{1}{\beta_+}$.

Recall that a strictly stable Lévy process with index $\alpha \in (0,2)$ satisfies Spitzer's condition for some parameter $\rho \in [0,1]$ and thus if $\rho \in (0,1)$ then

$$\mathbb{P}\left[\forall 0 \le t \le T : X_t \le 1\right] = T^{-\rho}.$$

Theorem (Aurzada/K./Lifshits'12+)

Let X be a strictly stable Lévy process with index $\alpha \in (0,2)$ and Spitzer's parameter $\rho \in (0,1)$. Then, we have for $\boldsymbol{\gamma < 1/\alpha}$

$$\mathbb{P}\left[\forall 0 \le t \le T : X_t \le \mathbf{1} + \mathbf{c}t^{\gamma}\right] = T^{-\rho+o(1)}$$

and

$$\mathbb{P}\left[\forall 0 \le t \le T : X_t \le \mathbf{1} - \mathbf{c}t^{\gamma}\right] = T^{-\rho+o(1)}.$$

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Remark:

The Theorem is also proved for Lévy processes belonging to the domain of attraction of a strictly stable Lévy processes with index $\alpha \in (0,2)$.

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The survival exponent stays $\rho \in (0, 1)$ (Spitzer's parameter) if

for general Lévy processes:

1. decreasing boundaries: $f(t) = 1 - t^{\gamma}$, $\gamma < \frac{1}{2}$,
   proved by Aurzada/K./Savov'12+
2. increasing boundaries: $f(t) = 1 + t^{\gamma}$, $\gamma < \max\{\frac{1}{2}, \rho\}$,
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Recall $\rho \leq \frac{1}{\alpha}$.

Thank you for your attention!

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