Symmetry breaking in quasi-1D Coulomb systems

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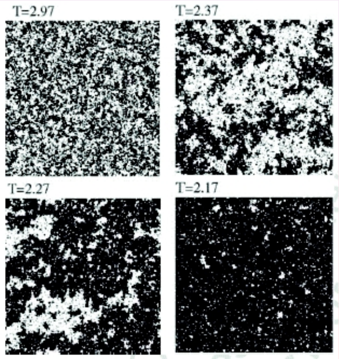
Symmetry breaking according to (adapted from) Wikipedia

Symmetry breaking is a phenomenon where fluctuations in a system at criticality determine which branch of a furcation is taken. The transition takes the system from a disorderly state into a more ordered state.



The jellium model

- Model was introduced by Wigner(38); C. Herring called it the *jellium*; also known as the *1-component plasma*.
- Consists of L electrons in a neutralizing uniform positively charged background.
- The thermodynamic limit exists ($L \to \infty$), and in 1D, symmetry breaking is known: Kunz(74), Brascamp-Lieb(75), Aizenman-Martin(80), Aizenman-Goldstein-Lebowitz(01).
- Roughly, particles split up into neutral "cells" leading to a one-parameter family of Gibbs equilibrium states.
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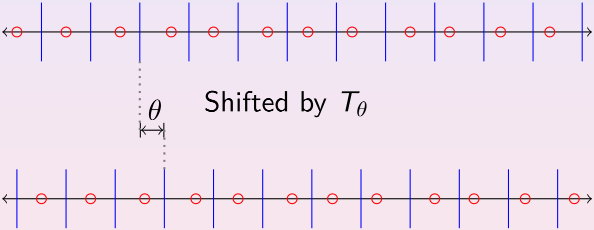
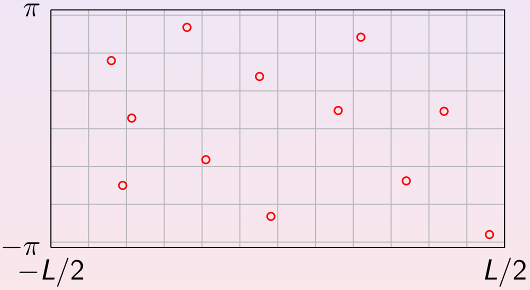
The 1D Wigner lattice



The quasi-one dimensional case

We consider charges in $[-\pi, \pi] \times \mathbb{R}$



AGL theorem

Theorem (Aizenman, Goldstein, Lebowitz)

Let $\nu(d\omega)$ be a translation-invariant point process on $\mathbb{R}$ with $\mathcal{N}_I(\omega)$ the number of points in I. If

$$\lim_{|I|\to\infty} \mathrm{Var}(\mathcal{N}_I) < \infty\,,$$

then ν is not mixing, and has a decomposition

$$\nu = \int_0^1 \nu_\theta \; d\theta\,,$$

into mutually singular measures.

Connection between $\mathcal{N}_I$ and the electric field

- The number of particles up to x, starting from the left, has a connection with the electric field at x:

$$\boxed{E(x) = x - \mathcal{N}_{[-L/2,x]}(\omega)}$$

- One can think of $E(x)$ as the charge imbalance at x.
- Aizenman-Martin(80) showed that $E(0)$ has a bounded variance as $L \to \infty$.

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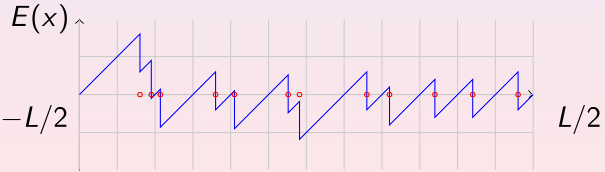
The charge imbalance function and total energy

The total energy U is given by

$$\frac{1}{2}\int E(x)^2 dx.$$

*The Gibbs measure $\frac{1}{Z}e^{-\beta U(\omega)}$.



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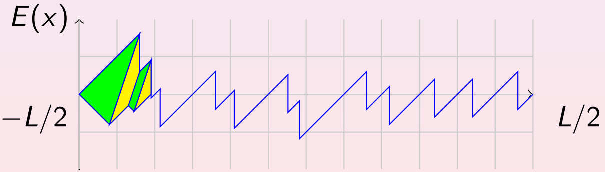
A picture of the total energy



Symmetry breaking via tight fluctuations

To prepare for the quasi-1D case, we will give a picture of

Bounded Variance in 1D

$\mathbb{P}(E(0) > c\lambda)$ decays like $e^{-c\lambda^3}$ as $L \to \infty$ and thus

$$\lim_{L\to\infty} \mathrm{Var}(E(0)) < \infty.$$

- The probability bound comes from analyzing $\frac{1}{Z}e^{-\beta U}$ using the *"Markov property"* for the electric field:

$$U_A + U_B = U_{A\cup B}.$$

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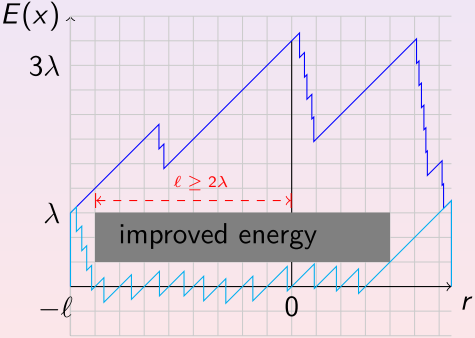
Finite strip energy difference estimate

We compare a high energy event with a low energy event:



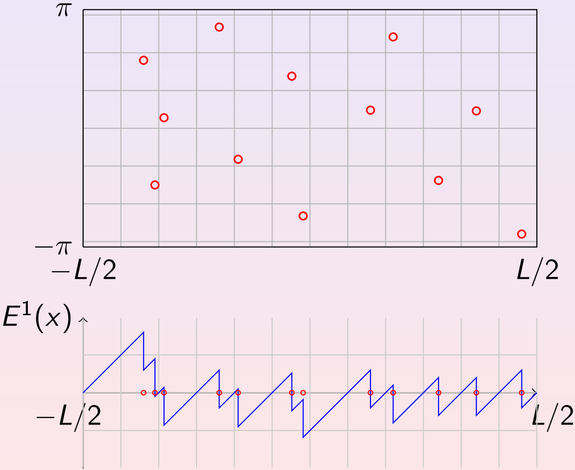
Energy of the quasi-1D system

Energy via a potential decomposition

$$U(\omega) = \frac{1}{2}\int_{\mathbb{L}} |E^1(x;\omega)|^2\, dx + \sum_{1\leq j<k\leq L} V_2(z_j - z_k)$$

The energy of the quasi-1D system differs from that of its 1D projection only in the V_2-component.

The quasi-1D charge imbalance function, $E^1(x)$



Two complications from V_2-interactions

Markov property

Instead of $U_{A\cup B} = U_A + U_B$ we get:

$$U_{A\cup B}(\omega) = U_A(\omega_A) + U_B(\omega_B) + \sum V_2(\omega_A, \omega_B).$$

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The energy gap for high and low energy configs is now bounded below by a 1D part plus V_2-energy for $-\ell < x_i, x_j < r$, which may be negative.

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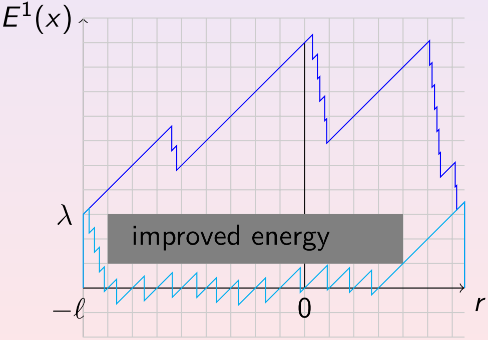

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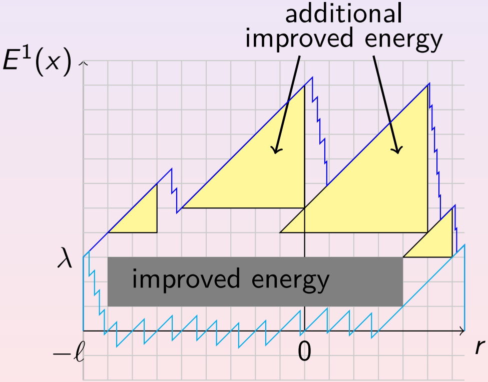
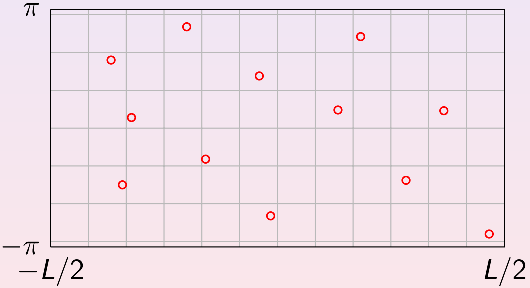


Rod replacements

We replace *bad particles* with *rods* similar to *smearing* charges introduced by Onsager(39) (we use rods instead of balls):



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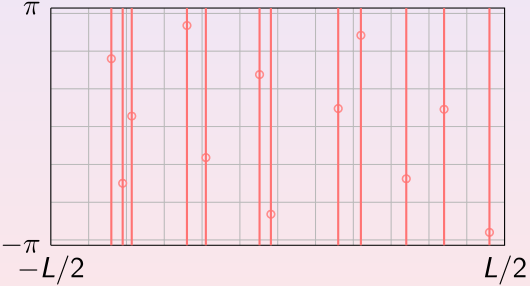
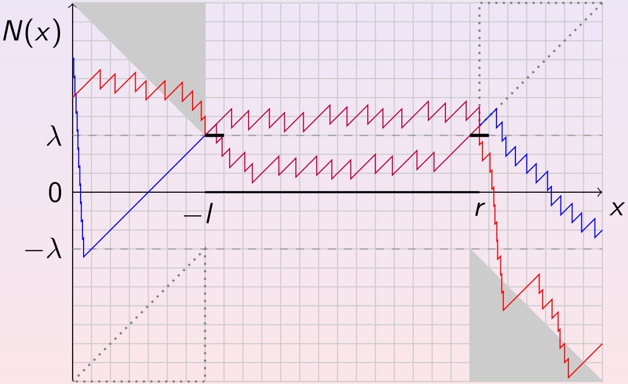


An additional energy change by moving the rod



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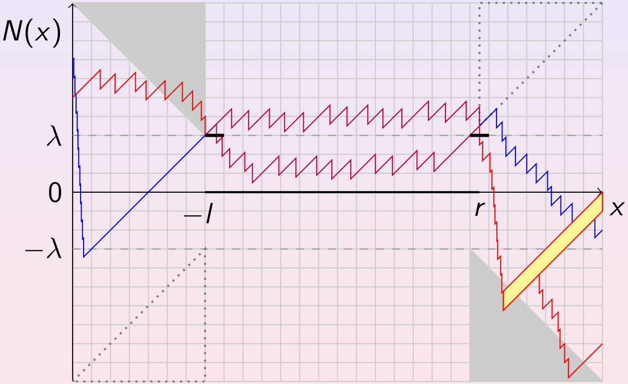

Rod replacements are not bijective!

Problem: A rod replacement couples *many* bad configurations to just *one* good one.

Solution: Let $\hat{B}$ be the nonempty set of indices of *bad particles*

$$\begin{aligned}\mathbb{P}\left(\text{bad}\right) &= \sum_{\hat{B}\subset\mathbb{N}} \mathbb{P}\left(\{B(\omega)=\hat{B}\}\right)\\ &\leq \sum_{\hat{B}\subset\mathbb{N}} C' e^{-\sum_{k\in\hat{B}}(Ck^2)}\mathbb{P}\left(\text{good}\right)\\ &\leq C'\left(\prod_{k\in\mathbb{N}}(1+e^{-Ck^2})\right)\mathbb{P}\left(\text{good}\right).\end{aligned}$$

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