Malliavin calculus and convergence in density

Yaozhong Hu

Department of Mathematics
The University of Kansas

Thursday June 7, 2012
at NSF/CBMS Conference
University of Alabama in Huntsville

This is a joint ongoing work with

Fei Lu

David Nualart

Outline

1. Motivation
2. (nonlinear) Wiener functionals
3. Malliavin calculus
4. Main results
5. Applications

1. Motivation

Central limit theorem:

Let $X_1, \cdots, X_n$ be independent, identically distributed random variables with mean m and variance σ^2.

$$\sqrt{n}\left(\frac{X_1 + \cdots + X_n}{n} - m\right) \longrightarrow N(0, \sigma^2)$$

$$\frac{X_1 + \cdots + X_n}{n} - m \approx \frac{\xi}{\sqrt{n}}, \quad \text{where } \xi \sim N(0, \sigma^2).$$

The above convergence is in the sense of distribution

$$F_n \to N(0, \sigma^2) \quad \text{in distribution}$$

$$P(F_n \le a) \to \int_{-\infty}^{a} \phi_\sigma(x) dx \quad \forall \ a \in \mathbb{R}$$

where $\phi_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$.

Other examples of multiple Itô integral F_n

$$F_n = \int_{[0,T]^q} f_n(t_1, \cdots, t_q) dB_{t_1} \cdots dB_{t_q} ,$$

where q is an fixed positive integer, $(B_t, t \geq 0)$ is a standard Brownian motion, f_n is a sequence of deterministic functions such that

$$\int_{[0,T]^q} f_n^2(t_1, \cdots, t_q) dt_1 \cdots dt_q$$

Convergence in density of multiple integrals

Are there $f_n(x)$ such that

$$P(F_n \leq a) = \int_{-\infty}^{a} f_n(x)dx$$

and

$$f_n(x) \longrightarrow \phi_\sigma(x)\,?$$

Tool: Malliavin calculus

2. (Nonlinear) Wiener functionals

$\Omega = C_0([0, T], \mathbb{R})$ = The set of all continuous functions ω starting at 0 ($\omega(0) = 0$).

It is a Banach space with the sup norm $\|\omega\| = \sup_{0 \le t \le T} |\omega(t)|$.

$\mathcal{F}$ be the σ-algebra generated by the open sets

P is the canonical Wiener measure on $(\Omega, \mathcal{F})$ such that $B_t : \Omega \to \mathbb{R}$ defined by $B_t(\omega) = \omega(t)$ is the standard Brownian motion.

A functional from $\Omega \to \mathbb{R}$ is called a Wiener functional.

Example

1. B_t 2. $\int_0^T |B_t|^p dt$

3. $\sup_{0 \le t \le T} |B_t|$

4. $I_{\{\sup_{0 \le t \le T} |B_t|\}}$

5. $\int_0^T f(t)dB_t$, where $f : [0, T] \to \mathbb{R}$ s.t. $\int_0^T f^2(t)dt < \infty$

6. multiple Itô-Wiener integral
$I_n(f_n) = \int_{[0,T]^n} f_n(t_1, \cdots, t_n)dB_{t_1} \cdots dB_{t_n}$, where $f_n : [0, T]^n \to \mathbb{R}$ is symmetric and $\displaystyle\int_{[0,T]^n} f_n^2(t_1, \cdots, t_n)dt_1 \cdots dt_n < \infty$.

7. $x_{t_0}, dx_t = b(x_t)dt + \sigma(x_t)dB_t$.

8. Functionals of the form $F = f(\int_0^T h_1(t)dB_t, ..., \int_0^T h_n(t)dB_t)$ is dense in $L^2(\Omega, \mathcal{F}, P)$,

where f can be the sets of all polynomials, smooth functions of polynomial growth, smooth functions of compact supports

$h_1, h_2, \cdots, h_n, \cdots$ is ONB of $L^2([0, T])$

Itô-Wiener's chaos expansion theorem:

Any $F \in L^2(\Omega, \mathcal{F}, P)$ can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n)\,,$$

where

$$f_n \in L^2([0,T]^n) \quad \text{and} \quad I_n(f_n) = \int_{[0,T]^n} f_n(t_1, \cdots, t_n) dB_{t_1} \cdots dB_{t_n}\,.$$

Exercises: 1. Find the chaos expansion for $I_{\{\sup_{0\le t\le} |B_t| \le \varepsilon\}}$

2. Find the chaos expansion of x_t, where
$dx_t = b(x_t)dt + \sigma(x_t)dB_t\,, x_0 = x$.

Analysis of functionals $F : \Omega \to \mathbb{R}$

Nonlinear functional analysis Gateaux derivatives, Frechet derivatives etc

Zeidler, E. Nonlinear functional analysis and its applications. I. Fixed-point theorems. Springer, 1986. xxi+897 pp.

Zeidler, E. Nonlinear functional analysis and its applications. II/A. Linear monotone operators. Springer, 1990. xviii+467 pp

Zeidler, E. Nonlinear functional analysis and its applications. II/B. Nonlinear monotone operators. Springer, 1990. pp. i-xvi and 469-1202.

Zeidler, E. Nonlinear functional analysis and its applications. III. Variational methods and optimization. Springer, 1985. xxii+662 pp.

Zeidler, E. Nonlinear functional analysis and its applications. IV. Applications to mathematical physics. Springer, 1988. xxiv+975

Nonlinear functional analysis on a Banach space with a measure (infinite dimensional harmonic analysis)

Gaussian measure (Lebesgue measure does not exist in infinite dimensions)

Why Malliavin derivative?

x_{t_0}, $dx_t = b(x_t)dt + \sigma(x_t)dB_t$.

$x_{t_0} : \Omega \to \mathbb{R}^d$ is not continuous.

Example: $\int_0^T \left(B_t^2 dB_t^1 - B_t^2 dB_t^1\right)$

Malliavin, P.

Stochastic calculus of variation and hypoelliptic operators. Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976), pp. 195-263, Wiley, New York-Chichester-Brisbane, 1978.

3. Malliavin derivative

Let $(B_t; t \geq 0)$ be a standard Brownian motion.

Given $F = f(\int_0^T h_1(t)dB_t, ..., \int_0^T h_n(t)dB_t)$, where $h_1, h_2, \cdots, h_n, \cdots$ are continuous functions of t and constitute an orthonormal basis of $L^2([0, T])$

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\int_0^T h_1(t)dB_t, ..., \int_0^T h_n(t)dB_t)h_i(t).$$

The derivative operator D is a closable and unbounded operator

$$\|DF\|_{1,p}^{p} = E(|F|^{p}) + E\left(\int_0^T |D_tF|^2\,dt\right)^{p/2}$$

Higher order derivatives

$\|DF\|_{k,p}$

$\mathbb{D}^{k,p}$

If $F = I_q(f_q)$, then

$$D_t F = \sum_{q=1}^{\infty} q I_q(f_q(\cdot, t)).$$

If $F = \sup_{0 \le t \le T} B_t$, then

$$D_t F = I_{[0,\theta_T]}(t)\,,$$

where θ_T is the unqiue maximum point of B_t over $[0, T]$

chain rule, $D_t g(F) = g'(F) D_t F$

Malliavin calculus can be developed for general Gaussian processes, for Poisson processes, Lévy processes

$H = L^2([0, T])$

Denote by δ the adjoint operator of D, characterized by the following duality relation:

$$E(\delta(u)F) = E(\langle DF, u\rangle_H) \quad \text{for any } F \in \mathbb{D}_{1,2}.$$

The operator δ is called the *divergence* operator.

Example

If $f \in L^2([0, T])$, then $\delta(h) = \int_0^T h(t)dB_t$

For $F = f(\int_0^T h_1(t)dB_t, ..., \int_0^T h_n(t)dB_t)$, where $h_1, h_2, \cdots, h_n, \cdots$ is an Orthonormal basis of $L^2([0, T])$, f is C^∞ with compact support.

Write

$$h = \alpha_1 h_1 + \cdots + \alpha_n h_n + \tilde{h}.$$

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T h(t)dB_t F\right] &= \mathbb{E}\Bigg[\left(\sum_{i=1}^n \alpha_i \int_0^T h_i(t)dB_t + \int_0^T \tilde{h}(t)dB_t\right) \\
&\quad f\left(\int_0^T h_1(t)dB_t, ..., \int_0^T h_n(t)dB_t\right)\Bigg] \\
&= (2\pi)^{-n/2}\sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} x_i f(x_1, \cdots, x_n) e^{-\frac{|x|^2}{2}} dx \\
&= -(2\pi)^{-n/2}\sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} f(x_1, \cdots, x_n) \frac{\partial}{\partial x_i} e^{-\frac{|x|^2}{2}} dx \\
&= (2\pi)^{-n/2}\sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} f(x_1, \cdots, x_n) e^{-\frac{|x|^2}{2}} dx \\
&= \mathbb{E}\left[\langle DF, h\rangle_H\right].
\end{aligned}
$$

Ornstein-Uhlenbeck operator

$$\delta DF = -LF.$$

Meyer's inequality

$$c_p \|F\|_{k,p} \leq \|(I+L)^{k/2} F\|_p \leq C_p \|F\|_{k,p} \,.$$

Interpolation inequality (Decreusefond-Hu-Üstünel)

For all $1 \leq p < \infty$, we have

$$\|(I+L)^{1/2}F\|_p \leq \frac{2}{\Gamma(1/2)}\|F\|_p^{1/2}\|(I+L)V\|_p^{1/2}.$$

Combined with Meyer's inequality

$$\|\nabla F\|_p \leq C_p(\|F\|_p + \|F\|_p^{1/2}\|\nabla^2 F\|_p^{1/2})$$

Lemma

$$\|\delta(u)\|_{L^p(\Omega)} \leq C_p \left(\|Eu\|_H + \|Du\|_{L^p(\Omega, H \otimes H)} \right).$$

Lemma

Let F be a random variable in the space $\mathbb{D}^{1,2}$ and suppose that $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of the operator δ in $L^2(\Omega)$. Then the law of F has a continuous and bounded density given by

$$p(x) = E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right].$$

Proof

$$
\begin{aligned}
p(x) &= \int_{\mathbb{R}} \delta_x(y)p(y)dy = E(\delta_x(F)) \\
&= E\left(\frac{d}{dy} \mathbf{1}_{\{y \geq x\}} \Big|_{y=F} \right) \\
&= E\left[\langle D\left(\mathbf{1}_{\{F>x\}}\right), DF\rangle_H \frac{1}{\|DF\|_H^2} \right] \\
&= E\left[\mathbf{1}_{\{F>x\}} \delta\left(\frac{DF}{\|DF\|_H^2} \right) \right].
\end{aligned}
$$

Another formula

$$\begin{aligned} p(x) &= E\left(\frac{d}{dy}\mathbf{1}_{\{y\geq x\}}\Big|_{y=F}\right) \\ &= E\left[\left\langle D\left(\mathbf{1}_{\{F>x\}}\right), u\right\rangle_H \frac{1}{\langle DF, u\rangle_H}\right] \\ &= E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{u}{\langle DF, u\rangle_H}\right)\right]. \end{aligned}$$

Nualart, D.

The Malliavin calculus and related topics, 2nd edition.

Springer (2006)

For any smooth function of compact support g

$$\begin{aligned}
& \int_{\mathbb{R}} g(x) E\left[\mathbf{1}_{\{F>x\}} \delta\left(\frac{u}{\langle DF, u\rangle_H}\right)\right] dx \\
= \; & E\left[\int_{-\infty}^{F} g(x) dx \delta\left(\frac{u}{\langle DF, u\rangle_H}\right)\right] \\
= \; & E\left[\langle D \int_{-\infty}^{F} g(x) dx, \frac{u}{\langle DF, u\rangle_H}\rangle_H\right] \\
= \; & E\left[\langle g(F) DF, \frac{u}{\langle DF, u\rangle_H}\rangle_H\right] \\
= \; & \mathbb{E}\left[g(F)\right]
\end{aligned}$$

We need more

Since $E\delta(u) = 0$

Lemma

Let F be a random variable and let $u \in \mathbb{D}^{1,q}(H)$ with $q > 1$. Then for the conjugate pair p and q (i.e. $\frac{1}{p} + \frac{1}{q} = 1$),

$$\left| E\left[\mathbf{1}_{\{F>x\}}\delta(u)\right]\right| \leq \left(P(|F| > |x|)\right)^{\frac{1}{p}} \|\delta(u)\|_{L^q(\Omega)} .$$

Denote

$$w = \|DF\|^2\,,\; u = \frac{DF}{w},\; v = \frac{-LF}{w}.$$

$$G_0 = 1\,, \quad G_{k+1} = \delta(G_k u)$$

Lemma

For any integer $m \geq 1$ *and any real number* $p > 1$*. Let* F *be a random variable such that* $F \in D^{m,p}$ *and* $E\|DF\|_H^{-p} < \infty$*. Then,* F *has a density* f *of class* C^∞*. Moreover,*

$$f_F^{(k)}(x) = (-1)^k E[\mathbf{1}_{\{F>x\}} G_{k+1}] \, .$$

$$
\begin{aligned}
G_0 &= 1 \\
G_1 &= \delta_u \\
G_2 &= \delta_u^2 - D_u\delta_u \\
G_3 &= \delta_u^3 - 3\delta_u D_u\delta_u + D_u^2\delta_u \\
G_4 &= \delta_u^4 - 6\delta_u^2 D_u\delta_u + 4\delta_u D_u^2\delta_u \\
&\quad - D_u^3\delta_u + 3\left(D_u\delta_u\right)^2 \\
G_5 &= \delta_u^5 - 10\delta_u^3 D_u\delta_u + 2\delta_u^2 D_u^2\delta_u - 5\delta_u D_u^3\delta_u \\
&\quad + 15\delta_u\left(D_u\delta_u\right)^2 + D_u^4\delta_u - 10 D_u\delta_u D_u^2\delta_u
\end{aligned}
$$

Lemma

Fix an integer m. Suppose $u \in L^2(\Omega, H)$ such that $D_u^k \delta_u^m \in L^2(\Omega)$, for $k = 0, 1, 2, \ldots, m$. (For example, $u \in \mathbb{D}^{m,2m}(H)$, since $E\delta_u^2 \leq \|u\|^2_{\mathbb{D}^{1,2,H}}$). Then we can recursively define a sequence $\{G_k\}_{k=0}^m$ by $G_0 = 1$ and $G_{k+1} = \delta(G_k u)$. Moreover, for $k = 1, 2, \ldots, m$, we can write G_k as

$$G_k = \sum_{i=0}^{[k/2]} c_{k,i} \delta_u^{k-2i} (D_u \delta_u)^i + HODT,$$

where we denote by *HODT* (the Higher order derivative terms) the sum of terms with derivatives of order bigger than 2, that is,

$$
\begin{aligned}
HODT \quad &= \sum_{\substack{i_0+i_1+\cdots+i_{k-1}\le k-1,\\ i_l\ge 0,\ i_2+\cdots+i_{k-1}\ge 1}} a_{i_0,i_1,\cdots,i_{k-1}} \delta_u^{i_0} \\
&\quad (D_u\delta_u)^{i_1} \left(D_u^2\delta_u\right)^{i_2} \cdots \left(D_u^{k-1}\delta_u\right)^{i_{k-1}} .
\end{aligned}
$$

3. Main results

Theorem

The following are equivalent:

(i) $\lim_{n\to\infty} \mathbb{E}[F_n^4] = 3$,

(ii) *For all* $1 \leq r \leq q-1$, $\lim_{n\to\infty} \|f_n \otimes_r f_n\|_{H^{\otimes 2(q-r)}} = 0$,

(iii) $\|DF_n\|_H^2 \to p$ *in* $L^2(\Omega)$ *as* $n \to \infty$.

(iv) F_n *converges in distribution to the normal law* $N(0,1)$ *as* $n \to \infty$.

Nualart, David; Peccati, Giovanni.

Central limit theorems for sequences of multiple stochastic integrals.

Ann. Probab. 33 (2005), no. 1, 177-93.

Nualart, D.; Ortiz-Latorre, S.

Central limit theorems for multiple stochastic integrals and Malliavin calculus.

Stochastic Process. Appl. 118 (2008), no. 4, 614-628.

Theorem (Hu-Nualart 05)

Let $F_k = \sum_{n=1}^{\infty} I_n(f_{n,k})$. *Suppose that*

- $\lim_{N\to\infty} \limsup_{k\to\infty} \sum_{n=N+1}^{\infty} n! \left\| f_{n,k} \right\|_{H^{\otimes n}}^2 = 0$;
- *for every* $n \geq 1$, $\lim_{k\to\infty} n! \left\| f_{n,k} \right\|_{H^{\otimes n}}^2 = \sigma_n^2$;
- $\sum_{n=1}^{\infty} \sigma_n^2 = \sigma^2 < \infty$;
- *for all* $n \geq 2$, $p = 1, ..., n-1$,
  $\lim_{k\to\infty} \left\| f_{n,k} \otimes_p f_{n,k} \right\|_{H^{\otimes 2(n-p)}}^2 = 0$.

Then, $F_k \longrightarrow N(0, \sigma^2)$ *as* k *tends to infinity.*

Hu, Y. Nualart, D.

Renormalized self-intersection local time for fractional Brownian motion.

Ann. Probab. 33 (2005), no. 3, 948–983.

Main result

Theorem (Hu-Lu-Nualart)

Let $\{F_n = I_q(f_n)\}_{n\in\mathbb{N}}$ *be in the qth Wiener chaos such that*

$$E[F_n^2] \to 1, \text{ as } n \to \infty, \tag{1}$$

and

$$\lim_{n\to\infty} \mathbb{E}\left|\|DF_n\|_H^2 - q\right|^2 \to 0. \tag{2}$$

Suppose $\sup_n \mathbb{E}[\|DF_n\|_H^{-8}] < \infty$*. Then, the density* $f_{F_n}(x)$ *of each* F_n *exists* $P(F_n \le a) = \int_{-\infty}^{a} f_{F_n}(x)dx \quad \forall\, a \in \mathbb{R}$ *and for any* $p \ge 1$,

$$\int_{\mathbb{R}} |f_{F_n}(x) - \phi(x)|^p dx \to 0\,.$$

Theorem (Hu-Lu-Nualart)

Let $\{F_n = I_q(f_n)\}_{n\in\mathbb{N}}$ satisfy the conditions (1)-(2) of previous theorem. Suppose that

$$\sup_n \mathbb{E}[\|DF_n\|_H^{-m}] < \infty \,. \tag{3}$$

Then the density $f_{F_n}(x)$ of F_n is smooth, and for any $k \geq 0$

$$\int_{\mathbb{R}} |f_{F_n}^{(k)}(x) - \phi^{(k)}(x)|^p dx \to 0 \,.$$

4. Applications

To verify the existence of negative moments

Watanabe, S.; Bismut, J.M.; Stroock, D.; Üstünel, A.S.; ...

Norris lemma (based on approach of Meyer, P.A.)

Small ball techniques (Kuelbs, James; Li, Wenbo; Shao Qiman; Chen Xia; ...)

$$\begin{aligned}\mathbb{E}(V^{-p}) &= \sum_{n=2}^{\infty}\mathbb{E}(V^{-p}I_{\{\frac{1}{n}\le V<\frac{1}{n-1}\}})+\mathbb{E}(V^{-p}I_{\{V\ge 1\}})\\ &\le 1+\sum_{n=2}^{\infty}n^pP(V<\frac{1}{n-1})\le 1+\sum_{n=2}^{\infty}n^p\left(\frac{1}{n-1}\right)^m<\infty\,.\end{aligned}$$

New task: Need uniform estimate

Theorem (Hu-Lu-Nualart)

Let $F_T = I_2(f_T)$ *with* $f_T = \sum_{i=1}^{\infty} \lambda_i^T e_i^T \otimes e_i^T$,.

Assume that λ_i^T *satisfies*

(i) $\lim_{T\to\infty} \sum_{i=1}^{\infty} \left(\lambda_i^T\right)^2 = \lim_{T\to\infty} \|f_T\|_{H^{\odot 2}}^2 = \frac{\sigma^2}{2}$;

(ii) $\lim_{T\to\infty} \sum_{i=1}^{\infty} \left(\lambda_i^T\right)^4 = 0$;

(iii) $\exists\ \varepsilon_0 > 0$ *s.t. for each* $T \in (0, \infty)$, *there exists an integer* $n = n(T) \geq 4p + 2$ *so that* $\sqrt{n}\left|\lambda_n^T\right| \geq 2\varepsilon_0$.

Then, each F_T *admits a density* $f_{F_T} \in C(\mathbb{R})$ *and*

$$\sup_{x\in\mathbb{R}} \left|f_{F_T}(x) - \phi(x)\right| \leq C_{p,\sigma,\varepsilon_0}[(\sum_{i=1}^{\infty} \left(\lambda_i^T\right)^4)^{\frac{1}{2}} + \left|EF_T^2 - \sigma^2\right|].$$

Hoffmann-Jøgensen, J.; Shepp, L. A.; Dudley, R. M.

On the lower tail of Gaussian seminorms.

Ann. Probab. 7 (1979), no. 2, 319-342.

Example 2

Fractional Ornstein-Uhlenbeck process

$$dX_t = -\theta X_t dt + \sigma dB_t^H, \quad X_0 \quad \text{is given}$$

where B_t^H is a fractional Brownian motion of Hurst parameter H.

Assume that H and σ are known, and we can continuously observe X_t. We want to estimate θ.

The least squares estmator is studied

Hu, Y. Nualart, D.

Parameter estimation for fractional Ornstein-Uhlenbeck processes.

Statist. Probab. Lett. 80 (2010), 1030-1038.

The least squares estimator

$$\widehat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \sigma \frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt}$$

Theorem (Hu, Nualart 2010)

Suppose $H \in \left[\frac{1}{2}, \frac{3}{4}\right)$. Then

$$\widehat{\theta}_T \to \theta \quad \text{almost surely}$$

$$\sqrt{T}\left[\widehat{\theta}_T - \theta\right] \xrightarrow{\mathcal{L}} N(0, \theta\sigma_H^2) \quad \text{(in distribution)}$$

$$\sigma_H^2 = (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)\Gamma(2H)}\right).$$

Proof

$$\widehat{\theta}_T = \theta - \sigma \frac{\int_0^T X_t dB_t^H / T}{\int_0^T X_t^2 dt / T}$$

It is proved

$$\frac{\int_0^T X_t^2 dt}{T} \to \sigma^2 \theta^{-2H} H\Gamma(2H) \quad \text{almost surely}$$

$$\frac{\int_0^T X_t dB_t^H}{T} \to 0 \quad \text{almost surely} .$$

This implies

$$\widehat{\theta}_T \to \theta$$

It is also proved

$$\frac{\int_0^T X_t dB_t^H}{\sqrt{T}} \xrightarrow{\mathcal{L}} N\left(0, \theta^{1-4H}\sigma^4\delta_H\right),$$

where

$$\begin{aligned}\delta_H &= H^2(4H-1)(\Gamma(2H)^2 \\ &\quad + \frac{\Gamma(2H)\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)})\end{aligned}$$

which implies

$$\widehat{\theta}_T \to \theta$$

Theorem

Let $\{F_n, n \geq 1\}$ be a sequence of random variables in the p-th Wiener chaos, $p \geq 2$, such that $\lim_{n\to\infty} \mathbb{E}(F_n^2) = \sigma^2$. Then the following conditions are equivalent:

(i) *F_n converges in law to $N(0, \sigma^2)$ as n tends to infinity.*

(ii) *$\|DF_n\|_{\mathcal{H}}^2$ converges in L^2 to a constant as n tends to infinity.*

$$\frac{\int_0^T X_t dB_t^H}{\sqrt{T}} \overset{\text{in density}}{\rightarrow} N\left(0, \theta^{1-4H}\sigma^4\delta_H\right),$$

$$f_T(t,s) = \frac{\sigma^2}{2\sqrt{T}} e^{-\theta|t-s|}.$$

Find the eigenvalues of the integral operator associated with the above kernel.

Open problems:

$$\sqrt{T}\left[\widehat{\theta}_T - \theta\right] \overset{\text{in density}}{\rightarrow} N(0, \theta\sigma_H^2)$$

$$\sqrt{T}\left[\widehat{\theta}_T - \theta\right] = \sigma \frac{\frac{\int_0^T X_t dB_t^H}{\sqrt{T}}}{\frac{\int_0^T X_t^2 dt}{T}}$$

THANK YOU