Malliavin calculus and convergence in density

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This is a joint ongoing work with

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Outline

1. Motivation
2. (nonlinear) Wiener functionals
3. Malliavin calculus
4. Main results
5. Applications
1. Motivation

Central limit theorem:

Let $X_1, \cdots, X_n$ be independent, identically distributed random variables with mean $m$ and variance $\sigma^2$.

$$\sqrt{n} \left( \frac{X_1 + \cdots + X_n}{n} - m \right) \rightarrow N(0, \sigma^2)$$

$$\frac{X_1 + \cdots + X_n}{n} - m \approx \frac{\xi}{\sqrt{n}}, \quad \text{where} \quad \xi \sim N(0, \sigma^2).$$

The above convergence is in the sense of distribution

$$F_n \rightarrow N(0, \sigma^2) \quad \text{in distribution}$$

$$P(F_n \leq a) \rightarrow \int_{-\infty}^{a} \phi_{\sigma}(x)dx \quad \forall \ a \in \mathbb{R}$$

where \( \phi_{\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} \).
Other examples of multiple Itô integral $F_n$

$$F_n = \int_{[0,T]^q} f_n(t_1, \cdots, t_q) dB_{t_1} \cdots dB_{t_q},$$

where $q$ is a fixed positive integer, $(B_t, t \geq 0)$ is a standard Brownian motion, $f_n$ is a sequence of deterministic functions such that

$$\int_{[0,T]^q} f_n^2(t_1, \cdots, t_q) dt_1 \cdots dt_q$$
Convergence in density of multiple integrals

Are there $f_n(x)$ such that

$$P(F_n \leq a) = \int_{-\infty}^{a} f_n(x) \, dx$$

and

$$f_n(x) \rightarrow \phi_{\sigma}(x)?$$

Tool: Malliavin calculus
2. (Nonlinear) Wiener functionals

\[ \Omega = C_0([0, T], \mathbb{R}) = \text{The set of all continuous functions } \omega \text{ starting at 0 (} \omega(0) = 0). \]

It is a Banach space with the sup norm \( \| \omega \| = \sup_{0 \leq t \leq T} |\omega(t)|. \)

\( \mathcal{F} \) be the \( \sigma \)-algebra generated by the open sets

\( P \) is the canonical Wiener measure on \( (\Omega, \mathcal{F}) \) such that

\( B_t : \Omega \to \mathbb{R} \) defined by \( B_t(\omega) = \omega(t) \) is the standard Brownian motion.

A functional from \( \Omega \to \mathbb{R} \) is called a Wiener functional.

Example

1. \( B_t \)
2. \( \int_0^T |B_t|^p \, dt \)
3. \( \sup_{0 \leq t \leq T} |B_t| \)
4. \( I_{\{\sup_{0 \leq t \leq T} |B_t| \}} \)
5. \( \int_0^T f(t) dB_t \), where \( f : [0, T] \to \mathbb{R} \) s.t. \( \int_0^T f^2(t) dt < \infty \)

6. multiple Itô-Wiener integral
\[
I_n(f_n) = \int_{[0,T]^n} f_n(t_1, \cdots, t_n) dB_{t_1} \cdots dB_{t_n}, \text{ where } f_n : [0, T]^n \to \mathbb{R}
\]
is symmetric and \( \int_{[0,T]^n} f_n^2(t_1, \cdots, t_n) dt_1 \cdots dt_n < \infty \).

7. \( x_t_0, dx_t = b(x_t) dt + \sigma(x_t) dB_t \).

8. Functionals of the form \( F = f(\int_0^T h_1(t) dB_t, \ldots, \int_0^T h_n(t) dB_t) \) is dense in \( L^2(\Omega, \mathcal{F}, P) \),

where \( f \) can be the sets of all polynomials, smooth functions of polynomial growth, smooth functions of compact supports

\( h_1, h_2, \cdots, h_n, \cdots \) is ONB of \( L^2([0, T]) \)
Itô-Wiener's chaos expansion theorem:

Any $F \in L^2(\Omega, \mathcal{F}, P)$ can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where

$$f_n \in L^2([0, T]^n) \quad \text{and} \quad I_n(f_n) = \int_{[0,T]^n} f_n(t_1, \cdots, t_n) dB_{t_1} \cdots dB_{t_n}.$$

Exercises: 1. Find the chaos expansion for $I_{\{\sup_{0 \leq t \leq |B_t| \leq \varepsilon}\}}$

2. Find the chaos expansion of $x_t$, where

$$dx_t = b(x_t)dt + \sigma(x_t)dB_t, \ x_0 = x.$$
Analysis of functionals $F : \Omega \to \mathbb{R}$

Nonlinear functional analysis Gateaux derivatives, Frechet derivatives etc


Nonlinear functional analysis on a Banach space with a measure (infinite dimensional harmonic analysis)

Gaussian measure (Lebesgue measure does not exist in infinite dimensions)
Why Malliavin derivative?

\[ x_{t_0}, dx_t = b(x_t) dt + \sigma(x_t) dB_t. \]

\( x_{t_0} : \Omega \to \mathbb{R}^d \) is not continuous.

Example: \( \int_0^T (B^2_t dB^1_t - B^2_t dB^1_t) \)
Malliavin, P.

3. Malliavin derivative

Let \((B_t; t \geq 0)\) be a standard Brownian motion.

Given \(F = f(\int_0^T h_1(t) dB_t, ..., \int_0^T h_n(t) dB_t)\), where \(h_1, h_2, \cdots, h_n, \cdots\) are continuous functions of \(t\) and constitute an orthonormal basis of \(L^2([0, T])\)

\[
D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\int_0^T h_1(t) dB_t, ..., \int_0^T h_n(t) dB_t)h_i(t).
\]

The derivative operator \(D\) is a closable and unbounded operator.
\[ \| DF \|_{1,p}^p = E(\| F \|^p) + E \left( \int_0^T |D_t F|^2 \, dt \right)^{p/2} \]

Higher order derivatives

\[ \| DF \|_{k,p} \]

\[ D_{k,p} \]
If $F = l_q(f_q)$, then

$$D_t F = \sum_{q=1}^{\infty} q l_q(f_q(\cdot, t)).$$

If $F = \sup_{0 \leq t \leq T} B_t$, then

$$D_t F = l_{[0, \theta_T]}(t),$$

where $\theta_T$ is the unique maximum point of $B_t$ over $[0, T]$. 

Chain rule, $D_t g(F) = g'(F)D_t F$
Malliavin calculus can be developed for general Gaussian processes, for Poisson processes, Lévy processes
\( H = L^2([0, T]) \)

Denote by \( \delta \) the adjoint operator of \( D \), characterized by the following duality relation:

\[
E(\delta(u)F) = E(\langle DF, u \rangle_H) \quad \text{for any } F \in D_{1,2}.
\]

The operator \( \delta \) is called the \textit{divergence} operator.

\textbf{Example}

If \( f \in L^2([0, T]) \), then \( \delta(h) = \int_0^T h(t)dB_t \)

For \( F = f(\int_0^T h_1(t)dB_t, \ldots, \int_0^T h_n(t)dB_t) \), where \( h_1, h_2, \ldots, h_n, \ldots \) is an Orthonormal basis of \( L^2([0, T]) \), \( f \) is \( C^\infty \) with compact support.

Write

\[
h = \alpha_1 h_1 + \cdots + \alpha_n h_n + \tilde{h}.
\]
\[
\mathbb{E} \left[ \int_0^T h(t) dB_t F \right] = \mathbb{E} \left[ \left( \sum_{i=1}^n \alpha_i \int_0^T h_i(t) dB_t + \int_0^T \tilde{h}(t) dB_t \right) \right.
\]

\[
\left. f \left( \int_0^T h_1(t) dB_t, \ldots, \int_0^T h_n(t) dB_t \right) \right]
\]

\[
= (2\pi)^{-n/2} \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} x_i f(x_1, \ldots, x_n) e^{-\frac{|x|^2}{2}} \, dx
\]

\[
= -(2\pi)^{-n/2} \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \frac{\partial}{\partial x_i} e^{-\frac{|x|^2}{2}} \, dx
\]

\[
= (2\pi)^{-n/2} \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) e^{-\frac{|x|^2}{2}} \, dx
\]

\[
= \mathbb{E} [\langle DF, h \rangle_H].
\]
Ornstein-Uhlenbeck operator

\[ \delta DF = -LF. \]
Meyer’s inequality

$$c_p \| F \|_{k,p} \leq \| (I + L)^{k/2} F \|_p \leq C_p \| F \|_{k,p}.$$
Interpolation inequality (Decreusefond-Hu-Üstünel)

For all $1 \leq p < \infty$, we have

$$
\|(I + L)^{1/2} F\|_p \leq \frac{2}{\Gamma(1/2)} \|F\|_p^{1/2} \|(I + L) V\|_p^{1/2}.
$$

Combined with Meyer’s inequality

$$
\|\nabla F\|_p \leq C_p (\|F\|_p + \|F\|_p^{1/2} \|\nabla^2 F\|_p^{1/2})
$$
Lemma

\[ \| \delta(u) \|_{L^p(\Omega)} \leq C_p \left( \| Eu \|_H + \| Du \|_{L^p(\Omega, H \otimes H)} \right). \]
Lemma

Let $F$ be a random variable in the space $\mathbb{D}^{1,2}$ and suppose that $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of the operator $\delta$ in $L^2(\Omega)$. Then the law of $F$ has a continuous and bounded density given by

$$p(x) = E \left[ 1_{\{F > x\}} \delta \left( \frac{DF}{\|DF\|_H^2} \right) \right].$$
Proof

\[ p(x) = \int_{\mathbb{R}} \delta_x(y)p(y)\,dy = E(\delta_x(F)) \]

\[ = E \left( \frac{d}{dy} \mathbf{1}_{\{y \geq x\}} \big|_{y=F} \right) \]

\[ = E \left[ \langle D \left( \mathbf{1}_{\{F>x\}} \right), DF \rangle_H \frac{1}{\|DF\|_H^2} \right] \]

\[ = E \left[ \mathbf{1}_{\{F>x\}} \delta \left( \frac{DF}{\|DF\|_H^2} \right) \right]. \]
Another formula

\[ p(x) = E \left( \frac{d}{dy} \mathbf{1}_{y \geq x} \bigg| y = F \right) \]

\[ = E \left[ \langle D \left( \mathbf{1}_{F > x} \right), u \rangle_H \frac{1}{\langle DF, u \rangle_H} \right] \]

\[ = E \left[ \mathbf{1}_{F > x} \delta \left( \frac{u}{\langle DF, u \rangle_H} \right) \right]. \]
Nualart, D.
The Malliavin calculus and related topics, 2nd edition.
Springer (2006)
For any smooth function of compact support $g$

\[
\int_{\mathbb{R}} g(x) E \left[ 1_{\{F > x\}} \delta \left( \frac{u}{\langle DF, u \rangle_H} \right) \right] \, dx
\]

\[
= E \left[ \int_{-\infty}^{F} g(x) \, dx \delta \left( \frac{u}{\langle DF, u \rangle_H} \right) \right]
\]

\[
= E \left[ \langle D \int_{-\infty}^{F} g(x) \, dx, \frac{u}{\langle DF, u \rangle_H} \rangle_H \right]
\]

\[
= E \left[ \langle g(F)DF, \frac{u}{\langle DF, u \rangle_H} \rangle_H \right]
\]

\[
= \mathbb{E} [g(F)]
\]
We need more

Since $E\delta(u) = 0$

**Lemma**

Let $F$ be a random variable and let $u \in \mathbb{D}^{1,q}(H)$ with $q > 1$. Then for the conjugate pair $p$ and $q$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$),

$$|E[\mathbf{1}_{\{F > x\}}\delta(u)]| \leq (P(|F| > |x|))^{\frac{1}{p}} \|\delta(u)\|_{L^q(\Omega)}.$$
Denote

\[ w = \|DF\|^2, \quad u = \frac{DF}{w}, \quad v = \frac{-LF}{w}. \]

\[ G_0 = 1, \quad G_{k+1} = \delta(G_k u) \]
Lemma

For any integer $m \geq 1$ and any real number $p > 1$. Let $F$ be a random variable such that $F \in D_{m,p}$ and $E \|DF\|_H^{-p} < \infty$. Then, $F$ has a density $f$ of class $C^\infty$. Moreover,

$$f_F^{(k)}(x) = (-1)^k E[1_{\{F>x\}} G_{k+1}] .$$
\[ G_0 = 1 \]
\[ G_1 = \delta_u \]
\[ G_2 = \delta_u^2 - D_u \delta_u \]
\[ G_3 = \delta_u^3 - 3\delta_u D_u \delta_u + D_u^2 \delta_u \]
\[ G_4 = \delta_u^4 - 6\delta_u^2 D_u \delta_u + 4\delta_u D_u^2 \delta_u \]
\[ -D_u^3 \delta_u + 3(D_u \delta_u)^2 \]
\[ G_5 = \delta_u^5 - 10\delta_u^3 D_u \delta_u + 2\delta_u^2 D_u^2 \delta_u - 5\delta_u D_u^3 \delta_u \]
\[ + 15\delta_u (D_u \delta_u)^2 + D_u^4 \delta_u - 10D_u \delta_u D_u^2 \delta_u \]
Lemma

Fix an integer $m$. Suppose $u \in L^2(\Omega, H)$ such that $D^k u \delta^m u \in L^2(\Omega)$, for $k = 0, 1, 2, \ldots, m$. (For example, $u \in \mathbb{D}^{m,2m}(H)$, since $E \delta^2 u \leq \|u\|_{\mathbb{D}^{1,2,H}}^2$.) Then we can recursively define a sequence $\{G_k\}_{k=0}^m$ by $G_0 = 1$ and $G_{k+1} = \delta(G_k u)$. Moreover, for $k = 1, 2, \ldots, m$, we can write $G_k$ as
\[ G_k = \sum_{i=0}^{[k/2]} c_{k,i} \delta_u^{k-2i} (D_u \delta_u)^i + HODT, \]

where we denote by \( HODT \) (the Higher order derivative terms) the sum of terms with derivatives of order bigger than 2, that is,
\[
HODT = \sum_{i_0 + i_1 + \cdots + i_{k-1} \leq k-1, \quad i_1 \geq 0, \ i_2 + \cdots + i_{k-1} \geq 1} a_{i_0, i_1, \ldots, i_{k-1}} \delta_u^{i_0} \\
(D_u \delta_u)^{i_1} (D_u^2 \delta_u)^{i_2} \cdots (D_u^{k-1} \delta_u)^{i_{k-1}}.
\]
3. Main results

Theorem

The following are equivalent:

(i) $\lim_{n \to \infty} \mathbb{E}[F_n^4] = 3$,

(ii) For all $1 \leq r \leq q - 1$, $\lim_{n \to \infty} \| f_n \otimes r f_n \|_{\mathcal{H}^{2(q-r)}} = 0$,

(iii) $\| DF_n \|_{\mathcal{H}}^2 \to p$ in $L^2(\Omega)$ as $n \to \infty$.

(iv) $F_n$ converges in distribution to the normal law $N(0, 1)$ as $n \to \infty$. 
Nualart, David; Peccati, Giovanni.
Central limit theorems for sequences of multiple stochastic integrals.

Nualart, D.; Ortiz-Latorre, S.
Central limit theorems for multiple stochastic integrals and Malliavin calculus.
Theorem (Hu-Nualart 05)

Let $F_k = \sum_{n=1}^{\infty} I_n(f_{n,k})$. Suppose that

- $\lim_{N \to \infty} \limsup_{k \to \infty} \sum_{n=N+1}^{\infty} n! \|f_{n,k}\|_{H \otimes n}^2 = 0$;
- for every $n \geq 1$, $\lim_{k \to \infty} n! \|f_{n,k}\|_{H \otimes n}^2 = \sigma_n^2$;
- $\sum_{n=1}^{\infty} \sigma_n^2 = \sigma^2 < \infty$;
- for all $n \geq 2$, $p = 1, \ldots, n-1$,
  \[ \lim_{k \to \infty} \left\| f_{n,k} \otimes_p f_{n,k} \right\|_{H \otimes 2(n-p)}^2 = 0. \]

Then, $F_k \longrightarrow N(0, \sigma^2)$ as $k$ tends to infinity.
Hu, Y. Nualart, D.

Renormalized self-intersection local time for fractional Brownian motion.

Main result

Theorem (Hu-Lu-Nualart)

Let \( \{ F_n = l_q(f_n) \}_{n \in \mathbb{N}} \) be in the \( q \)th Wiener chaos such that

\[
E[F_n^2] \to 1, \text{ as } n \to \infty,
\]

and

\[
\lim_{n \to \infty} E \left| \| DF_n \|_H^2 - q \right|^2 \to 0.
\]

Suppose \( \sup_n E[\| DF_n \|_H^8] < \infty \). Then, the density \( f_{F_n}(x) \) of each \( F_n \) exists \( P(F_n \leq a) = \int_{-\infty}^{a} f_{F_n}(x)dx \) \( \forall a \in \mathbb{R} \) and for any \( p \geq 1 \),

\[
\int_{\mathbb{R}} |f_{F_n}(x) - \phi(x)|^p dx \to 0.
\]
Theorem (Hu-Lu-Nualart)

Let \( \{ F_n = l_q(f_n) \} \in \mathbb{N} \) satisfy the conditions (1)-(2) of previous theorem. Suppose that

\[
\sup_n \mathbb{E}[\|DF_n\|^{-m}] < \infty.
\]

Then the density \( f_{F_n}(x) \) of \( F_n \) is smooth, and for any \( k \geq 0 \)

\[
\int_{\mathbb{R}} |f_{F_n}^{(k)}(x) - \phi^{(k)}(x)|^p dx \to 0.
\]
4. Applications

To verify the existence of negative moments

Watanabe, S.; Bismut, J.M.; Stroock, D.; Üstünel, A.S.; ...

Norris lemma (based on approach of Meyer, P.A.)

Small ball techniques (Kuelbs, James; Li, Wenbo; Shao Qiman; Chen Xia; ...)

\[
\mathbb{E}(V^{-p}) = \sum_{n=2}^{\infty} \mathbb{E}(V^{-p}I_{\{\frac{1}{n} \leq V < \frac{1}{n-1}\}}) + \mathbb{E}(V^{-p}I_{\{V \geq 1\}})
\]

\[
\leq 1 + \sum_{n=2}^{\infty} n^p P(V < \frac{1}{n-1}) \leq 1 + \sum_{n=2}^{\infty} n^p \left(\frac{1}{n-1}\right)^m < \infty.
\]

New task: Need uniform estimate
Theorem (Hu-Lu-Nualart)

Let $F_T = l_2(f_T)$ with $f_T = \sum_{i=1}^{\infty} \lambda_i^T e_i^T \otimes e_i^T$.

Assume that $\lambda_i^T$ satisfies

(i) $\lim_{T \to \infty} \sum_{i=1}^{\infty} (\lambda_i^T)^2 = \lim_{T \to \infty} \|f_T\|^2_{H^\otimes_2} = \frac{\sigma^2}{2}$;

(ii) $\lim_{T \to \infty} \sum_{i=1}^{\infty} (\lambda_i^T)^4 = 0$;

(iii) $\exists \varepsilon_0 > 0$ s.t. for each $T \in (0, \infty)$, there exists an integer $n = n(T) \geq 4p + 2$ so that $\sqrt{n} |\lambda_n^T| \geq 2\varepsilon_0$.

Then, each $F_T$ admits a density $f_{F_T} \in C(\mathbb{R})$ and

$$\sup_{x \in \mathbb{R}} |f_{F_T}(x) - \phi(x)| \leq C_{p, \sigma, \varepsilon_0} \left[ \left( \sum_{i=1}^{\infty} (\lambda_i^T)^4 \right)^{\frac{1}{2}} + \left| E F_T^2 - \sigma^2 \right| \right].$$
Hoffmann-Jøgensen, J.; Shepp, L. A.; Dudley, R. M.
On the lower tail of Gaussian seminorms.
Example 2

Fractional Ornstein-Uhlenbeck process

\[ dX_t = -\theta X_t dt + \sigma dB_t^H, \quad X_0 \quad \text{is given} \]

where \( B_t^H \) is a fractional Brownian motion of Hurst parameter \( H \).

Assume that \( H \) and \( \sigma \) are known, and we can continuously observe \( X_t \). We want to estimate \( \theta \).

The least squares estimator is studied

Hu, Y. Nualart, D.

Parameter estimation for fractional Ornstein-Uhlenbeck processes.

The least squares estimator

\[
\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \sigma \frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt}
\]

Theorem (Hu, Nualart 2010)

Suppose \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right) \). Then

\[
\hat{\theta}_T \to \theta \text{ almost surely}
\]

\[
\sqrt{T} \left( \hat{\theta}_T - \theta \right) \xrightarrow{\mathcal{D}} N(0, \theta \sigma_H^2) \quad \text{(in distribution)}
\]

\[
\sigma_H^2 = (4H - 1) \left( 1 + \frac{\Gamma(3 - 4H)\Gamma(4H - 1)}{\Gamma(2 - 2H)\Gamma(2H)} \right).
\]
Proof

\[ \hat{\theta}_T = \theta - \sigma \frac{\int_0^T X_t dB^H_t / T}{\int_0^T X_t^2 dt / T} \]

It is proved

\[ \frac{\int_0^T X_t^2 dt}{T} \to \sigma^2 \theta^{-2H} H \Gamma(2H) \text{ almost surely} \]

\[ \frac{\int_0^T X_t dB^H_t}{T} \to 0 \text{ almost surely.} \]

This implies

\[ \hat{\theta}_T \to \theta \]
It is also proved

\[
\int_0^T X_t d\mathcal{B}^H_t \xrightarrow{\mathcal{L}} N \left( 0, \theta^{1-4H} \sigma^4 \delta_H \right),
\]

where

\[
\delta_H = H^2 (4H - 1) \frac{\Gamma(2H)^2}{\Gamma(2 - 2H)} + \frac{\Gamma(2H) \Gamma(3 - 4H) \Gamma(4H - 1)}{\Gamma(2 - 2H)}
\]

which implies

\[
\hat{\theta}_T \to \theta
\]
Use Malliavin calculus

**Theorem**

Let \( \{ F_n, n \geq 1 \} \) be a sequence of random variables in the \( p \)-th Wiener chaos, \( p \geq 2 \), such that \( \lim_{n \to \infty} \mathbb{E}(F_n^2) = \sigma^2 \). Then the following conditions are equivalent:

(i) \( F_n \) converges in law to \( N(0, \sigma^2) \) as \( n \) tends to infinity.

(ii) \( \| DF_n \|_{L^2}^2 \) converges in \( L^2 \) to a constant as \( n \) tends to infinity.

\[
\int_0^T X_t dB_t^H \quad \text{in density} \quad \rightarrow \quad N\left(0, \theta^{1-4H} \sigma^4 \delta_H\right),
\]

\[
f_T(t, s) = \frac{\sigma^2}{2\sqrt{T}} e^{-\theta|t-s|}.
\]

Find the eigenvalues of the integral operator associated with the above kernel.
Open problems:

\[\sqrt{T} \left[ \hat{\theta}_T - \theta \right] \quad \xrightarrow{\text{in density}} \quad N(0, \theta \sigma_H^2)\]

\[\sqrt{T} \left[ \hat{\theta}_T - \theta \right] = \sigma \frac{\int_0^T X_t dB_t^H}{\sqrt{T}} \frac{\int_0^T X_t^2 dt}{T}\]
THANK YOU