Small value probabilities for continuous state branching processes with immigration

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Continuous state branching process

$Z = (Z_t : t \geq 0)$: defined on $(\Omega, \mathcal{F})$ with a family of probabilities $(\mathbb{P}_x, x \geq 0)$,

satisfying

- $[0, \infty)-$valued strong Markov process
- right continuous with left limit paths
- branching property: for any $\lambda, x, y \geq 0$,

$$\mathbb{P}_{x+y} e^{-\lambda Z_t} = \mathbb{P}_x e^{-\lambda Z_t} \mathbb{P}_y e^{-\lambda Z_t}, \tag{1.1}$$

Z with $Z_0 = x > 0$ is called a continuous state branching process starting from x.

Laplace transform of CSBP

$$\mathbb{P}_x e^{-\lambda Z_t} = e^{-x u_t(\lambda)}, \quad t \geq 0, \quad \lambda \geq 0, \quad x \geq 0, \tag{1.2}$$

where $u_t(\lambda)$ satisfies

$$u_0(\lambda) = \lambda, \qquad \frac{\partial}{\partial t} u_t(\lambda) + \psi(u_t(\lambda)) = 0, \tag{1.3}$$

and

$$\psi(\lambda) = -m\lambda + \alpha\lambda^2 + \int_0^\infty \left(e^{-\lambda x} - 1 + \lambda x\right) \Pi(dx), \tag{1.4}$$

with $m > 0$, $\alpha \geq 0$, and

$$\int_0^\infty \left(x \wedge x^2\right) \Pi(dx) < \infty. \tag{1.5}$$

$\mathbb{E}Z_t = e^{mt}$, $e^{-mt}Z_t$ is a positive martingale.

D. R. Grey [G74]

$$e^{-mt}Z_t \to W \ \mathbb{P}_x - a.s. \ \& \ L^1 \Leftrightarrow \int_1^{\infty} (x \log x)\Pi(dx) < \infty.$$

Define

$$\theta_t := \inf\{s > 0 : \int_0^s Z_u \mathrm{d}u > t\}$$
$$X_t := Z_{\theta_t}$$

then $(X_t, t \geq 0)$ is a Lévy process with $\log \mathbb{E} e^{-\lambda X_1} = \psi(\lambda)$. When

$$\psi(\lambda) = -a\lambda + \int_0^\infty (e^{-\lambda x} - 1)\Pi(\mathrm{d}x), \tag{1.6}$$

$(X_t, t \geq 0)$ is called a subordinator.

Small value probability of CSBP

定理

(Bingham [B76]) If ψ *is not corresponding to a subordinator and* $\int^{\infty} 1/\psi(\lambda)d\lambda < \infty$*, then*

$$\mathbb{P}(W = 0) = e^{-\gamma} \quad \text{with} \quad \gamma = \inf\{s \geq 0 : \psi(s) = 0\}. \quad (1.7)$$

Write $\rho = -\psi'(\gamma)/m$*, then*

$$\mathbb{P}(0 < W \leq x) \sim Cx^{\rho} \qquad \text{as } x \to 0+.$$

Assume ψ is corresponding to a subordinator.
(i) If ψ has zero drift and finite Lévy measure $\Pi(0,\infty) = \alpha m$, then

$$\mathbb{P}_1(W \le \varepsilon) \sim \varepsilon^{\alpha} L(1/\varepsilon) \qquad as\ \varepsilon \to 0^+$$

for some function L varying slowly as infinity. One can take L constant if and only if

$$\int_0^1 x^{-1}\Pi(\mathrm{d}x) < \infty.$$

(ii) If ψ has zero drift and infinity Lévy measure, then

$$-\log \mathbb{P}_1(W \le \varepsilon) \sim L^*(1/\varepsilon) \qquad as\ \varepsilon \to 0^+,$$

where L^* is a slowly varying function.
(iii) If ψ has drift $a > 0$, then

$$-\log \mathbb{P}_1(W \le \varepsilon) \sim \varepsilon^{-a/(m-a)} L(1/\varepsilon) \qquad as\ \varepsilon \to 0^+$$

for some function L slowly varying at infinity.

Continuous state branching process with immigration

$(\mathcal{Z}_t, \mathbb{P}_x : t \geq 0)$, the Laplace transform of $\mathcal{Z}$ is given by

$$\mathbb{E}_x e^{-\lambda \mathcal{Z}_t} = \exp\left\{-x u_t(\lambda) - \int_0^t \varphi(u_s(\lambda))ds\right\}. \tag{2.8}$$

where

$$\varphi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda x}\right) n(dx), \tag{2.9}$$

with $b \geq 0$, and

$$\int_0^\infty (1 \wedge x) n(dx) < \infty. \tag{2.10}$$

$\mathcal{Z}$ is called a continuous state branching process with immigration starting from $x \geq 0$, and denoted as $CBI(\psi, \varphi)$.

M. A. Pinsky (1972): Limit theorems for continuous state branching process with immigration. Bull. Amer. Math. Soci. **78**, 242-244.

定理

$e^{-mt}\mathcal{Z}_t$ *has a finite and non-degenerate limit denoted by* $\mathcal{W}$ *iff*

$$\int_1^\infty (x\log x)\Pi(dx) < \infty, \quad \int_1^\infty (\log x)n(dx) < \infty.$$

Tauberian Theorems

Assume V is a positive random variable.
(i)(Karamata Tauberian Theorem) For constants $C > 0$ and $\alpha > 0$ and a function L slowly varying at infinity,

$$\mathbb{E}e^{-\lambda V} \sim C\lambda^{-\alpha}L(\lambda) \qquad \lambda \to \infty,$$

if and only if

$$\mathbb{P}(V \leq t) \sim \frac{C}{\Gamma(1+\alpha)} t^{\alpha} L(1/t) \qquad t \to 0^{+}.$$

(ii)(de Bruijn's Tauberian Theorem) Assume $0 \leq \alpha < 1$ is a constant, L is a slowly varying function at infinity, and L^* is the conjugate slowly varying function to L defined in Bingham and Teugels [BT75]. Then

$$\log \mathbb{E}e^{-\lambda V} \sim -\lambda^{\alpha}/L(\lambda)^{1-\alpha} \qquad \lambda \to \infty, \tag{2.11}$$

if and only if

$$\log \mathbb{P}(V \leq t) \sim -(1-\alpha)\alpha^{\alpha/(1-\alpha)} t^{-\alpha/(1-\alpha)} L^*(t^{-1/(1-\alpha)}) \quad \text{as } t \to 0^+. \tag{2.12}$$

In particular, when $\alpha = 0$, then

$$\log \mathbb{E}e^{-\lambda V} \sim -1/L(\lambda) \quad \text{iff} \quad \log \mathbb{P}(V \leq t) \sim -L^*(t^{-1}).$$

Small value probability of $\mathcal{W}$

定理

If the branching mechanism ψ *is not corresponding to the Laplace exponent of a subordinator and* $\int^{\infty} 1/\psi(\lambda)d\lambda < \infty$, *then*

$$\mathbf{P}_x e^{-\lambda\mathcal{W}} = \exp\left\{-x\phi(\lambda) - \int_0^{\lambda}\frac{\varphi(\phi(t))}{t}dt\right\} \sim C\lambda^{-\tau'} \qquad as\ \lambda\to\infty,$$

for some constant $C > 0$ *that is independent of* λ, *with* $\tau' = \varphi(\gamma)/m$.

(i) If ψ has zero drift and finite Lévy measure $\Pi(0,\infty)=\alpha m$, then

$$\begin{aligned} -\log \mathbb{P}_x(\mathcal{W}\le\varepsilon) \quad \sim \quad & (2m)^{-1}b\alpha\cdot|\log\varepsilon|^2 \\ & +\alpha^{\beta}(m(\beta+1))^{-1}\mathbb{I}_{\{b=0\}}\cdot|\log\varepsilon|^{\beta+1}, \end{aligned}$$

(ii) If ψ has zero drift and infinity Lévy measure, then

$$\begin{aligned} -\log \mathbb{P}_x(\mathcal{W}\le\varepsilon) \quad \sim \quad & m^{-1}b\cdot R_1^*(1/\varepsilon) \\ & +\mathbb{I}_{\{b=0\}}\cdot\left(xL^*(1/\varepsilon)+m^{-1}\cdot R_2^*(1/\varepsilon)\right). \end{aligned}$$

(iii) If ψ has drift $a>0$ and the initial value $x>0$, then

$$-\log \mathbb{P}_x(\mathcal{W}\le\varepsilon) \quad \sim \quad (x+b/a)^{m/(m-a)}\cdot\varepsilon^{-a/(m-a)}L\left(1/\varepsilon\right)$$

(iv) If ψ has drift $a>0$ and the initial value $x=0$, then

$$\begin{aligned} -\log \mathbb{P}_0(\mathcal{W}\le\varepsilon) \quad \sim \quad & (b/a)^{m/(m-a)}\cdot\varepsilon^{-a/(m-a)}L\left(1/\varepsilon\right) \\ & +m^{-m/(m-a\beta)}(m-a\beta)(a\beta)^{-1}\mathbb{I}_{\{b=0\}} \\ & \cdot\varepsilon^{-a\beta/(m-a\beta)}L_2^*\left(\varepsilon^{-m/(m-a\beta)}\right) \end{aligned}$$

Thank you!

Reference

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