Some Results on Excursion Probability of Gaussian Random Fields

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1. Introduction and Motivation

Let $X = \{X(t), t \in T\}$ be a real-valued Gaussian random field, where T is the parameter set. For large $u > 0$, how to evaluate the excursion probability

$$\mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\} = ?$$

- Double sum method by Pickands (1967), Piterbarg (1996), Chan and Lai (2006).
- Tube method by Sun (1993).
- Rice method initiated by Rice (1944) and developed by many others: Adler, Azais and Wschebor, etc.
- Euler characteristic method by Worsley (1996), Taylor, Takemura and Adler (2005), Taylor and Adler (2007).

Euler characteristic method

- Let $A_u = \{t \in T : X(t) \geq u\}$ be the excursion set.
- Let $\varphi(A_u)$ and $\mathbb{E}\{\varphi(A_u)\}$ be the Euler characteristic and mean Euler characteristic of A_u respectively.
- In one dimension, the Euler characteristic is the number of connected components; in two dimensions, it is the number of connected components minus the number of holes.
- When $T = [0, 1]$, $\varphi(A_u)$ is like the number of upcrossings, whose expectation has been used to approximate $\mathbb{P}\{\sup_{0\leq t\leq 1} X(t) \geq u\}$ for a long time.

Theorem 1.1 [Taylor, Takemura and Adler (2005)]
Let $X = \{X(t), t \in T\}$ be a centered smooth Gaussian random field with **unit variance** , then there exists some $\alpha > 0$ such that

$$\mathbb{P}\left\{\sup_{t\in T} X(t) \geq u\right\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}), \text{ as } u \to \infty.$$

- $\mathbb{E}\{\varphi(A_u)\}$ is computable, see Adler and Taylor (2007),

  $$\mathbb{E}\{\varphi(A_u)\} = C_0\Psi(u) + \sum_{j=1}^{\dim(T)} C_j u^{j-1} e^{-u^2/2},$$

  where Ψ is the tail probability of standard Normal, C_j are constants depending on X and T.

Example 1.1 Let X be a smooth isotropic Gaussian field with unit variance and $T = [0, L]^N$, then

$$\mathbb{E}\{\varphi(A_u)\} = \Psi(u) + \sum_{j=1}^{N} \frac{\binom{N}{j} L^j \lambda^{j/2}}{(2\pi)^{(j+1)/2}} H_{j-1}(u) e^{-u^2/2},$$

where $\lambda = \mathrm{Var}(X_i(t))$ and $H_{j-1}(u)$ are Hermite polynomials.

- We use notations $\frac{\partial X(t)}{\partial t_i} = X_i(t)$ and $\frac{\partial^2 X(t)}{\partial t_i \partial t_j} = X_{ij}(t)$.

Question: The **constant-variance** condition is too restrictive for many applications. For Gaussian fields not having constant variance, how to compute the mean Euler characteristic or can it still be used to approximate the excursion probability?

- Consider rectangle $T = \prod_{i=1}^{N}[a_i, b_i]$.

Definition

A face J of dimension k, is defined by fixing a subset $\sigma(J) \subset \{1, \cdots, N\}$ of size k and a subset $\varepsilon(J) = \{\varepsilon_j, j \notin \sigma(J)\} \subset \{0, 1\}^{N-k}$ of size $N - k$, so that

$$\begin{aligned} J = \{t \in T : a_j < t_j < b_j \text{ if } j \in \sigma(J), \\ t_j = (1 - \varepsilon_j)a_j + \varepsilon_j b_j \text{ if } j \notin \sigma(J)\}. \end{aligned}$$

- Let $\partial_k T$ be the collection of faces of dimension k in T, then $\overset{\circ}{T} = \partial_N T$ and $\partial T = \cup_{k=0}^{N-1} \cup_{J \in \partial_k T} J$.

For $J \in \partial_k T$, define the number of extended outward maxima above level u by

$$\begin{aligned} M_u^E(J) \triangleq \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0,\\ \text{index}(\nabla^2 X_{|J}(t)) = k, \varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\}, \end{aligned}$$

where $\varepsilon_j^* = 2\varepsilon - 1$, and the index of a matrix is defined by the number of its negative eigenvalues.

Example: Let $T = [0, 1] = \{0\} \cup \{1\} \cup (0, 1)$, then

$$\begin{aligned} \left\{ \sup_{t \in T} X(t) \geq u \right\} = \{X(0) \geq u, X'(0) \leq 0\} \cup \{X(1) \geq u, X'(1) \geq 0\}\\ \cup \{\exists t \in (0, 1) \text{ s.t. } X(t) \geq u, X'(t) = 0, X''(t) < 0\}. \end{aligned}$$

For general case, since $T = \cup_{k=0}^{N} \cup_{J \in \partial_k T} J$,

$$\left\{ \sup_{t \in T} X(t) \geq u \right\} = \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} \{M_u^E(J) \geq 1\}.$$

Thus

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{P}\Big\{\bigcup_{k=0}^{N} \bigcup_{J\in\partial_k T} \{M_u^E(J) \geq 1\}\Big\}.$$

By Bonferroni inequality and Piterbarg (1996),

$$\begin{aligned}\sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)\} &\geq \mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \\ &\geq \sum_{k=0}^{N} \sum_{J\in\partial_k T} \Big(\mathbb{E}\{M_u^E(J)\} - \frac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}\Big) \\ &\quad - \sum_{J\neq J'} \mathbb{E}\{M_u^E(J)M_u^E(J')\}.\end{aligned}$$

- Let $\upsilon(t) := \mathrm{Var}(X(t))$, $\sup_{t\in T}\upsilon(t) = 1$. We call a function $h(u)$ super-exponentially small, if there exists $\alpha > 0$ such that $h(u) = o(e^{-\alpha u^2 - u^2/2})$ as $u \to \infty$.

Proposition 2.1 Under certain smooth and regular conditions, $\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}$ and $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ are super-exponentially small, thus there exists some $\alpha > 0$ such that

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)\} + o(e^{-\alpha u^2 - u^2/2}).$$

- Applying Kac-Rice formula,

$$\begin{aligned}
&\mathbb{E}\{M_u^E(J)\} \\
&= \int_J dt\, p_{\nabla X_{|J}(t)}(0)\mathbb{E}\{|\det\nabla^2 X_{|J}(t)| I(X(t)\geq u) I(\text{index}(\nabla^2 X_{|J}(t)) = k) \\
&\quad \times I(\varepsilon_j^* X_j(t) > 0 \text{ for all } j \notin \sigma(J)) | \nabla X_{|J}(t) = 0\}.
\end{aligned}$$

Theorem 2.1 [Morse's theorem, see Adler and Taylor (2007)]
Let X be a Morse function on rectangle T. Then

$$\varphi(A_u) = \sum_{k=0}^{N} \sum_{J\in\partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(J),$$

where $\mu_i(J)$ is the number of points in $t \in J$ satisfying

$$\begin{aligned} X(t) &\geq u, & \\ X_j(t) &= 0, & j \in \sigma(J), \\ \varepsilon_j^* X_j(t) &> 0, & j \notin \sigma(J), \\ \mathrm{index}(\nabla^2 X_{|J}(t)) &= i, & \end{aligned}$$

where $\varepsilon_j^* = 2\varepsilon - 1$, and the index of a matrix is defined by the number of its negative eigenvalues.

Theorem 2.2 Let T be rectangle. Then under certain smooth and regular conditions, there exists some $\alpha > 0$ such that

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \\
&= \sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)\} + o(e^{-\alpha u^2 - u^2/2}) \\
&= \sum_{k=0}^{N} \sum_{J\in\partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(J)\} + o(e^{-\alpha u^2 - u^2/2}) \\
&= \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}).
\end{aligned}$$

Remark The result is also valid for T being piecewise smooth convex set or manifold without boundary, sphere for example.

Examples for $T = [a, b] \in \mathbb{R}$

Example 2.1 Suppose b is the only point such that $\nu(b) = 1$. If $\nu'(b) \neq 0$, then

$$\mathbb{P}\left\{ \sup_{t\in[a,b]} X(t) \geq u \right\} \sim \Psi(u).$$

Example 2.2 [Laplace Method] Suppose $t_0 \in (a, b)$ is the only point such that $\nu(t_0) = 1$. Then

$$\mathbb{P}\left\{ \sup_{t\in[a,b]} X(t) \geq u \right\} \sim \left(\frac{\mathrm{Var}(X'(t_0))}{\mathbb{E}\{X(t_0)X''(t_0)\}} + 1 \right)^{-1/2} \Psi(u).$$

Example 2.3 Suppose b is the only point such that $\nu(b) = 1$. If $\nu'(b) = 0$, then

$$\mathbb{P}\Big\{\sup_{t\in[a,b]} X(t) \geq u\Big\} \sim \left(\frac{1}{2}\left(\frac{\mathrm{Var}(X'(t_0))}{\mathbb{E}\{X(t_0)X''(t_0)\}} + 1\right)^{-1/2} + \frac{1}{2}\right)\Psi(u).$$

Example 2.4 Let $T = [a_1, b_1] \times [a_2, b_2]$. Suppose $M = \{t : \nu(t) = 1\} = \{b_1\} \times [a_2, b_2]$ and $\nabla\nu(t) \neq 0$ for all $t \in M$, then

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \sim \mathbb{P}\Big\{\sup_{t\in M} X(t) \geq u\Big\}.$$

Theorem 3.1 [Pickands (1969), Qualls and Watanabe (1973)]
Let $\{X(t) : t \in D \subset \mathbb{R}^N\}$ be a stationary Gaussian field, where D is a bounded Jordan measurable set. If for a constant $\alpha \in (0, 2]$, as $\|s\| \to 0$,

$$\mathbb{E}\{X(t)X(t+s)\} = 1 - \|s\|^\alpha + o(\|s\|^\alpha), \tag{1}$$

then as $u \to \infty$,

$$\mathbb{P}\Big\{\sup_{t\in D} X(t) \geq u\Big\} \sim \mathrm{Vol}(D) H_\alpha u^{2N/\alpha} \Psi(u),$$

where H_α is Pickands' constant.

- Chan and Lai (2006) extends the result to nonstationary Gaussian or even asymptotic Gaussian with property similar to (1).

Let $r_t : \mathcal{S} = \{x \in [0,\infty)^N : \sum_{i=1}^N x_i = 1\} \to \mathbb{R}_+$ is a continuous function satisfying

$$\sup_{x\in\mathcal{S}} |r_t(x) - r_s(x)| \to 0, \quad \text{as } \|t-s\| \to 0.$$

Let $\alpha > 0$, $p = (p_1, \cdots, p_N)$ with $p_i > 0$ for all $1 \le i \le N$. We study the excursion probability of centered Gaussian field X satisfying

$$\begin{aligned}\mathbb{E}\{X(t)X(t+s)\} = 1 - (1+o(1))\Big(\sum_{i=1}^N |s_i|^{p_i}\Big)^{\alpha} L\Big(\sum_{i=1}^N |s_i|^{p_i}\Big) \\ \times r_t\Big(\frac{|s_1|^{p_1}}{\sum_{i=1}^N |s_i|^{p_i}}, \cdots, \frac{|s_N|^{p_N}}{\sum_{i=1}^N |s_i|^{p_i}}\Big),\end{aligned} \tag{2}$$

as $\|s\| \to 0$, uniformly over $t \in \bar{D}$, where L is a slowly varying function.

Let $\{W_t(x) : t, x \in [0,\infty)^N\}$ be a Gaussian random field such that $W_t(0) = 0$,

$$\mathbb{E}(W_t(x)) = -\left(\sum_{i=1}^N x_i^{p_i}\right)^\alpha r_t\left(\frac{x_1^{p_1}}{\sum_{i=1}^N x_i^{p_i}}, \cdots, \frac{x_N^{p_N}}{\sum_{i=1}^N x_i^{p_i}}\right),$$

$$\begin{aligned}
&\mathrm{Cov}(W_t(x), W_t(y)) \\
&= \left(\sum_{i=1}^N x_i^{p_i}\right)^\alpha r_t\left(\frac{x_1^{p_1}}{\sum_{i=1}^N x_i^{p_i}}, \cdots, \frac{x_N^{p_N}}{\sum_{i=1}^N x_i^{p_i}}\right) \\
&+ \left(\sum_{i=1}^N y_i^{p_i}\right)^\alpha r_t\left(\frac{y_1^{p_1}}{\sum_{i=1}^N y_i^{p_i}}, \cdots, \frac{y_N^{p_N}}{\sum_{i=1}^N y_i^{p_i}}\right) \\
&- \left(\sum_{i=1}^N |x_i - y_i|^{p_i}\right)^\alpha r_t\left(\frac{|x_1 - y_1|^{p_1}}{\sum_{i=1}^N |x_i - y_i|^{p_i}}, \cdots, \frac{|x_N - y_N|^{p_N}}{\sum_{i=1}^N |x_i - y_i|^{p_i}}\right).
\end{aligned}$$

Define

$$H(t) = \lim_{K\to\infty} K^{-N} \int_0^\infty e^{\lambda} \mathbb{P}\Big\{ \sup_{0\le x_i\le K, \forall i} W_t(x) > \lambda \Big\} d\lambda. \tag{3}$$

Theorem 3.2 Let $\{X(t) : t \in D \subset \mathbb{R}^N\}$ be a centered Gaussian field satisfying condition (2). Then under some mild regularity conditions, as $u \to \infty$,

$$\mathbb{P}\Big\{ \sup_{t\in D} X(t) > u \Big\} \sim \Psi(u) \left(\prod_{i=1}^{N} \Delta_{u,i}^{-1} \right) \int_D H(t)dt, \tag{4}$$

where

$$\Delta_{u,i} = \min\{x > 0 : x^{\alpha p_i} L(x) = u^{-2}\}, \qquad \forall 1 \le i \le N.$$

Remark This result still holds when X is asymptotically Gaussian.

Thank you