Some Results on Excursion Probability of Gaussian Random Fields

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(joint work with Yimin Xiao)

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1. Introduction and Motivation

Let $X = \{X(t), t \in T\}$ be a real-valued Gaussian random field, where $T$ is the parameter set. For large $u > 0$, how to evaluate the excursion probability

$$\mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\} = ?$$

- Tube method by Sun (1993).
- Rice method initiated by Rice (1944) and developed by many others: Adler, Azais and Wschebor, etc.
Euler characteristic method

- Let $A_u = \{ t \in T : X(t) \geq u \}$ be the excursion set.
- Let $\phi(A_u)$ and $\mathbb{E}\{\phi(A_u)\}$ be the Euler characteristic and mean Euler characteristic of $A_u$ respectively.
- In one dimension, the Euler characteristic is the number of connected components; in two dimensions, it is the number of connected components minus the number of holes.
- When $T = [0, 1]$, $\phi(A_u)$ is like the number of upcrossings, whose expectation has been used to approximate $\mathbb{P}\{\sup_{0 \leq t \leq 1} X(t) \geq u\}$ for a long time.
Theorem 1.1 [Taylor, Takemura and Adler (2005)]
Let $X = \{X(t), t \in T\}$ be a centered smooth Gaussian random field with unit variance, then there exists some $\alpha > 0$ such that

$$
\mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\} = \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}), \text{ as } u \to \infty.
$$

- $\mathbb{E}\{\varphi(A_u)\}$ is computable, see Adler and Taylor (2007),

$$
\mathbb{E}\{\varphi(A_u)\} = C_0 \Psi(u) + \sum_{j=1}^{\dim(T)} C_j u^{j-1} e^{-u^2/2},
$$

where $\Psi$ is the tail probability of standard Normal, $C_j$ are constants depending on $X$ and $T$. 
Example 1.1 Let $X$ be a smooth isotropic Gaussian field with unit variance and $T = [0, L]^N$, then

$$\mathbb{E}\{\varphi(A_u)\} = \Psi(u) + \sum_{j=1}^{N} \frac{\binom{N}{j} L^j \lambda^{j/2}}{(2\pi)^{(j+1)/2}} H_{j-1}(u) e^{-u^2/2},$$

where $\lambda = \text{Var}(X_i(t))$ and $H_{j-1}(u)$ are Hermite polynomials.

- We use notations $\frac{\partial X(t)}{\partial t_i} = X_i(t)$ and $\frac{\partial^2 X(t)}{\partial t_i \partial t_j} = X_{ij}(t)$. 
**Question:** The constant-variance condition is too restrictive for many applications. For Gaussian fields not having constant variance, how to compute the mean Euler characteristic or can it still be used to approximate the excursion probability?
Consider rectangle \( T = \prod_{i=1}^{N} [a_i, b_i] \).

**Definition**

A face \( J \) of dimension \( k \), is defined by fixing a subset \( \sigma(J) \subset \{1, \cdots, N\} \) of size \( k \) and a subset \( \varepsilon(J) = \{\varepsilon_j, j \notin \sigma(J)\} \subset \{0, 1\}^{N-k} \) of size \( N - k \), so that

\[
J = \{t \in T : a_j < t_j < b_j \text{ if } j \in \sigma(J),
\]

\[
t_j = (1 - \varepsilon_j)a_j + \varepsilon_jb_j \text{ if } j \notin \sigma(J) \}.
\]

Let \( \partial_k T \) be the collection of faces of dimension \( k \) in \( T \), then \( \partial N T = \partial T = \bigcup_{k=0}^{N-1} \bigcup_{J \in \partial_k T} J \).
For $J \in \partial_k T$, define the number of extended outward maxima above level $u$ by

$$M_u^E(J) \triangleq \#\{t \in J : X(t) \geq u, \nabla X|_J(t) = 0, \text{ index}(\nabla^2 X|_J(t)) = k, \varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\},$$

where $\varepsilon_j^* = 2\varepsilon - 1$, and the index of a matrix is defined by the number of its negative eigenvalues.

**Example:** Let $T = [0, 1] = \{0\} \cup \{1\} \cup (0, 1)$, then

$$\left\{ \sup_{t \in T} X(t) \geq u \right\} = \{X(0) \geq u, X'(0) \leq 0\} \cup \{X(1) \geq u, X'(1) \geq 0\}$$

$$\cup \{\exists t \in (0, 1) \text{ s.t. } X(t) \geq u, X'(t) = 0, X''(t) < 0\}.$$  

For general case, since $T = \bigcup_{k=0}^N \bigcup_{J \in \partial_k T} J$,

$$\left\{ \sup_{t \in T} X(t) \geq u \right\} = \bigcup_{k=0}^N \bigcup_{J \in \partial_k T} \{M_u^E(J) \geq 1\}.$$
Thus

\[ \mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\} = \mathbb{P}\left\{ \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} \left\{ M^E_u(J) \geq 1 \right\} \right\}. \]

By Bonferroni inequality and Piterbarg (1996),

\[
\sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M^E_u(J)\} \geq \mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\}
\geq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \left( \mathbb{E}\{M^E_u(J)\} - \frac{1}{2} \mathbb{E}\{M^E_u(J)(M^E_u(J) - 1)\} \right)
- \sum_{J \neq J'} \mathbb{E}\{M^E_u(J)M^E_u(J')\}.
\]

- Let \( \nu(t) := \text{Var}(X(t)) \), \( \sup_{t \in T} \nu(t) = 1 \). We call a function \( h(u) \) super-exponentially small, if there exists \( \alpha > 0 \) such that \( h(u) = o(e^{-\alpha u^2 - u^2/2}) \) as \( u \to \infty \).
Proposition 2.1 Under certain smooth and regular conditions, $\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}$ and $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ are super-exponentially small, thus there exists some $\alpha > 0$ such that

$$\mathbb{P}\left\{\sup_{t \in T} X(t) \geq u\right\} = \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\} + o(e^{-\alpha u^2-u^2/2}).$$

Applying Kac-Rice formula,

$$\mathbb{E}\{M_u^E(J)\} = \int_{J} dt \, p_{\nabla X|J(t)}(0) \mathbb{E}\{|\text{det}\nabla^2 X|J(t)|I(X(t) \geq u)I(\text{index}(\nabla^2 X|J(t)) = k)\} \times I(\varepsilon_j^* X_j(t) > 0 \text{ for all } j \notin \sigma(J)) |\nabla X|J(t) = 0\}. $$
Theorem 2.1 [Morse’s theorem, see Adler and Taylor (2007)]
Let $X$ be a Morse function on rectangle $T$. Then

$$
\varphi(A_u) = \sum_{k=0}^{N} \sum_{J \in \partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(J),
$$

where $\mu_i(J)$ is the number of points in $t \in J$ satisfying

- $X(t) \geq u$,
- $X_j(t) = 0$, $j \in \sigma(J)$,
- $\varepsilon_j^* X_j(t) > 0$, $j \notin \sigma(J)$,
- $\text{index}(\nabla^2 X_{|J}(t)) = i$,

where $\varepsilon_j^* = 2\varepsilon - 1$, and the index of a matrix is defined by the number of its negative eigenvalues.
**Theorem 2.2** Let $T$ be rectangle. Then under certain smooth and regular conditions, there exists some $\alpha > 0$ such that

$$
\mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\} = \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\} + o(e^{-\alpha u^2 - u^2/2})
$$

$$
= \sum_{k=0}^{N} \sum_{J \in \partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mathbb{E}\{\mu_i(J)\} + o(e^{-\alpha u^2 - u^2/2})
$$

$$
= \mathbb{E}\{\varphi(A_u)\} + o(e^{-\alpha u^2 - u^2/2}).
$$

**Remark** The result is also valid for $T$ being piecewise smooth convex set or manifold without boundary, sphere for example.
Example 2.1  Suppose $b$ is the only point such that $\nu(b) = 1$. If $\nu'(b) \neq 0$, then
\[
\mathbb{P}\left\{ \sup_{t \in [a,b]} X(t) \geq u \right\} \sim \Psi(u).
\]

Example 2.2  [Laplace Method]  Suppose $t_0 \in (a, b)$ is the only point such that $\nu(t_0) = 1$. Then
\[
\mathbb{P}\left\{ \sup_{t \in [a,b]} X(t) \geq u \right\} \sim \left( \frac{\text{Var}(X'(t_0))}{\mathbb{E}\{X(t_0)X''(t_0)\}} + 1 \right)^{-1/2} \Psi(u).
\]
Example 2.3 Suppose $b$ is the only point such that $\nu(b) = 1$. If $\nu'(b) = 0$, then

$$
\mathbb{P}\left\{ \sup_{t \in [a,b]} X(t) \geq u \right\}
\sim \left( \frac{1}{2} \left( \frac{\text{Var}(X'(t_0))}{\mathbb{E}\{X(t_0)X''(t_0)\}} + 1 \right)^{-1/2} + \frac{1}{2} \right) \Psi(u).
$$

Example 2.4 Let $T = [a_1, b_1] \times [a_2, b_2]$. Suppose $M = \{ t : \nu(t) = 1 \} = \{ b_1 \} \times [a_2, b_2]$ and $\nabla \nu(t) \neq 0$ for all $t \in M$, then

$$
\mathbb{P}\left\{ \sup_{t \in T} X(t) \geq u \right\} \sim \mathbb{P}\left\{ \sup_{t \in M} X(t) \geq u \right\}.
$$
3. Nonsmooth Anisotropic Gaussian Fields

**Theorem 3.1** [Pickands (1969), Qualls and Watanabe (1973)]

Let \( \{X(t) : t \in D \subset \mathbb{R}^N\} \) be a stationary Gaussian field, where \( D \) is a bounded Jordan measurable set. If for a constant \( \alpha \in (0, 2] \), as \( \|s\| \to 0 \),

\[
\mathbb{E}\{X(t)X(t + s)\} = 1 - \|s\|^\alpha + o(\|s\|^\alpha),
\]

then as \( u \to \infty \),

\[
\mathbb{P}\left\{ \sup_{t \in D} X(t) \geq u \right\} \sim \text{Vol}(D) H_\alpha u^{2N/\alpha} \Psi(u),
\]

where \( H_\alpha \) is Pickands’ constant.

- Chan and Lai (2006) extends the result to nonstationary Gaussian or even asymptotic Gaussian with property similar to (1).
Let $r_t : S = \{x \in [0, \infty)^N : \sum_{i=1}^{N} x_i = 1\} \to \mathbb{R}_+$ is a continuous function satisfying
\[ \sup_{x \in S} |r_t(x) - r_s(x)| \to 0, \text{ as } \|t - s\| \to 0. \]

Let $\alpha > 0$, $p = (p_1, \cdots, p_N)$ with $p_i > 0$ for all $1 \leq i \leq N$. We study the excursion probability of centered Gaussian field $X$ satisfying
\[
\mathbb{E}\{X(t)X(t + s)\} = 1 - (1 + o(1)) \left( \sum_{i=1}^{N} |s_i|^{p_i} \right)^{\alpha} L \left( \sum_{i=1}^{N} |s_i|^{p_i} \right)
\times r_t \left( \frac{|s_1|^{p_1}}{\sum_{i=1}^{N} |s_i|^{p_i}}, \cdots, \frac{|s_N|^{p_N}}{\sum_{i=1}^{N} |s_i|^{p_i}} \right),
\]

as $\|s\| \to 0$, uniformly over $t \in \bar{D}$, where $L$ is a slowly varying function.
Let \( \{ W_t(x) : t, x \in [0, \infty)^N \} \) be a Gaussian random field such that \( W_t(0) = 0 \),

\[
E(W_t(x)) = - \left( \sum_{i=1}^{N} x_i^{p_i} \right)^{\alpha} r_t \left( \frac{x_1^{p_1}}{\sum_{i=1}^{N} x_i^{p_i}}, \cdots, \frac{x_N^{p_N}}{\sum_{i=1}^{N} x_i^{p_i}} \right),
\]

\[
\text{Cov}(W_t(x), W_t(y))
\]

\[
= \left( \sum_{i=1}^{N} x_i^{p_i} \right)^{\alpha} r_t \left( \frac{x_1^{p_1}}{\sum_{i=1}^{N} x_i^{p_i}}, \cdots, \frac{x_N^{p_N}}{\sum_{i=1}^{N} x_i^{p_i}} \right)
\]

\[
+ \left( \sum_{i=1}^{N} y_i^{p_i} \right)^{\alpha} r_t \left( \frac{y_1^{p_1}}{\sum_{i=1}^{N} y_i^{p_i}}, \cdots, \frac{y_N^{p_N}}{\sum_{i=1}^{N} y_i^{p_i}} \right)
\]

\[
- \left( \sum_{i=1}^{N} |x_i - y_i|^{p_i} \right)^{\alpha} r_t \left( \frac{|x_1 - y_1|^{p_1}}{\sum_{i=1}^{N} |x_i - y_i|^{p_i}}, \cdots, \frac{|x_N - y_N|^{p_N}}{\sum_{i=1}^{N} |x_i - y_i|^{p_i}} \right).
\]
Define

\[ H(t) = \lim_{K \to \infty} K^{-N} \int_0^\infty e^\lambda \mathbb{P} \left\{ \sup_{0 \leq x_i \leq K, \forall i} W_t(x) > \lambda \right\} d\lambda. \]  

(3)

**Theorem 3.2** Let \( \{X(t) : t \in D \subset \mathbb{R}^N\} \) be a centered Gaussian field satisfying condition (2). Then under some mild regularity conditions, as \( u \to \infty \),

\[ \mathbb{P} \left\{ \sup_{t \in D} X(t) > u \right\} \sim \Psi(u) \left( \prod_{i=1}^N \Delta_{u,i}^{-1} \right) \int_D H(t) dt, \]  

(4)

where

\[ \Delta_{u,i} = \min\{x > 0 : x^{\alpha p_i} L(x) = u^{-2}\}, \quad \forall 1 \leq i \leq N. \]

**Remark** This result still holds when \( X \) is asymptotically Gaussian.
Thank you