Survival probabilities of weighted random walks

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Problem: Given a centered process \((Z_t)_{t \geq 0}\), determine asymptotics of

\[
p(T) := P \left[ \sup_{t \in [0,T]} Z_t \leq 1 \right], \quad T \to \infty.
\]

Typical: \(p(T) \asymp T^{-\theta}\) or \(p(T) = T^{-\theta+o(1)}\), \(\theta\) is called survival exponent.

\(f \asymp g\ (T \to \infty)\) means that \(f(t)/g(t)\) is bounded away from zero and infinity for all \(t\) large enough.

Related problem:

\[
P \left[ \sup_{t \in [0,1]} Z_t \leq \epsilon \right], \quad \epsilon \downarrow 0.
\]
Brownian motion: \( \theta = 1/2 \) (reflection principle):

\[
P \left[ \sup_{t \in [0,T]} B_t \leq 1 \right] = P[|B_T| \leq 1] \sim \sqrt{\frac{2}{\pi}} T^{-1/2}.
\]

Centered random walks \((S_n)_{n \geq 1}\) with finite variance: \( \theta = 1/2 \):

\[
P \left[ \sup_{n=1,\ldots,N} S_n \leq 1 \right] \sim c N^{-1/2}
\]
X_1, X_2, \ldots \text{ sequence of i.i.d. random variables with } E[X_1] = 0 \text{ and } E[X_1^2] = 1.

\( \sigma : [0, \infty) \to [0, \infty) \text{ some function with } \sigma(T) \to \infty \text{ as } T \to \infty. \)

Weighted random walk (WRW) \( Z = (Z_n)_{n \geq 1} \text{ defined as } \)

\[ \sum_{k=1}^{n} \sigma(k) X_k. \]

Goal: determine survival exponent of \( Z \) for a large class of functions \( \sigma. \)

\[ E[Z_n Z_m] = t_n \wedge t_m \text{ where } t_k = \sigma(1)^2 + \cdots + \sigma(k)^2. \text{ If } X_k \sim \mathcal{N}(0, 1) \text{ i.i.d.:} \]

\[ (Z_n)_{n \geq 1} \overset{d}{=} (B_{t_n})_{n \geq 1}. \]

Gaussian case: consider

\[ P \left[ \sup_{n=1,\ldots,N} B_{\kappa(n)} \leq 1 \right], \quad N \to \infty. \]
Theorem (Aurzada/B. 2011)

Assume that \( \kappa(N) \asymp N^q \) for some \( q > 0 \) and \( \kappa(N+1) - \kappa(N) \lesssim N^\delta \) for some \( \delta < q \). Then

\[
P \left[ \sup_{n=1,\ldots,N} B_{\kappa(n)} \leq 1 \right] = N^{-q/2 + o(1)}.
\]

In particular,

\[
P \left[ \sup_{n=1,\ldots,N} B_{\kappa(n)} \leq 1 \right] = P \left[ \sup_{t \in [0,\kappa(N)]} B_t \leq 1 \right] N^{o(1)}.
\]

Extension possible to functions \( \kappa(n) = \exp(n^\alpha) \) if \( 0 < \alpha < 1/4 \).
Sketch of proof

- Lower bound:
  \[ \{ B_t \leq 1, \forall t \in [0, \kappa(N)] \} \subseteq \{ B_{\kappa(n)} \leq 1, \forall n \leq N \} \]

- Upper bound: One can find \( \gamma \in (0, 1/2) \) and \( \alpha > 0 \) s.t.

\[
P \left[ \sup_{n=1,\ldots,N} B_{\kappa(n)} \leq 1 \right] \leq \left( \sup_{t \in [(\log N)^{\alpha}, \kappa(N)]} B_t - t^{\gamma} \leq 1 \right) + o(N^{-q/2}).
\]

- For \( c \in \mathbb{R} \) and \( \gamma < 1/2 \), it holds that (Uchiyama 1980)

\[
P \left[ \sup_{t \in [0,T]} B_t - ct^{\gamma} \leq 1 \right] \asymp T^{-1/2}, \quad T \to \infty,
\]

- Slepian’s inequality.
**Extension to WRW: Polynomial case**

- \( Z_n = \sum_{k=1}^{n} \sigma(k) X_k, \sigma(N) \asymp N^p. \)
- Corresponds to \( \kappa(n) = \sum_{k=1}^{n} \sigma(k)^2 \asymp n^{2p+1} \) in the Gaussian case. Moreover, \( \kappa(n+1) - \kappa(n) = \sigma(n+1)^2 \asymp n^{2p} \Rightarrow \) survival exponent is \( p + 1/2 \)

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**Theorem (Aurzada/B. 2011)**

Let \( (X_k)_{k \geq 1} \) be a sequence of i.i.d. centered random variables with \( E[X_1^2] = 1 \). Let \( \sigma(N) \asymp N^p \) for some \( p > 0 \). If \( E[|X_1|^\alpha] < \infty \) for some \( \alpha > 4p + 2 \), then

\[
P \left[ \sup_{n=1,\ldots,N} Z_n \leq 1 \right] \asymp N^{-(p+1/2)}, \quad N \to \infty.
\]
Idea of proof

- Apply a Skorokhod embedding to the martingale \((Z_n)_{n \geq 1}\): there is an increasing sequence of stopping times \((\tau(n))_{n \geq 0}\) and a Brownian motion s.t. \((B_{\tau(n)})_{n \geq 0} \overset{d}{=} (Z_n)_{n \geq 0}\) and \((B_{t \wedge \tau(n)})_{t \geq 0}\) is uniformly integrable for all \(n\).

\[
E[\tau(n)] = E\left[ B_{\tau(n)}^2 \right] = E[Z_n^2] = \sum_{k=1}^{n} \sigma(k)^2.
\]

\[
P\left[ \sup_{n=1,\ldots,N} Z_n \leq 1 \right] = P\left[ \sup_{n=1,\ldots,N} B_{\tau(n)} \leq 1 \right]
\]
Extension to WRW: Polynomial case

Theorem (Aurzada/B. 2011)

Let $(X_k)_{k \geq 1}$ be a sequence of i.i.d. centered random variables. Let $\sigma$ be increasing and $\sigma(N) \asymp N^p$ for some $p > 0$. If $E[e^{\alpha|X_1|}] < \infty$ for some $\alpha > 0$, then

$$P\left[ \sup_{n=1,\ldots,N} Z_n \leq 1 \right] \lesssim N^{-(p+1/2)+o(1)}, \quad N \to \infty.$$ 

- The proof relies on a coupling of Komlós, Major and Tusnády (1976) that allows to reduce the problem to the Gaussian case.

$$P\left[ \sup_{n=1,\ldots,N} \left| \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} \tilde{X}_k \right| \geq C \log N \right] \to 0.$$
Gaussian framework with exponential weight function

- Consider for $\beta > 0$

$$P \left[ \sup_{n=0,\ldots,N} B(e^{\beta n}) \leq 0 \right] = P \left[ \sup_{n=0,\ldots,N} U_{\beta n} \leq 0 \right].$$

- Recall that $U = (e^{-t/2}B(e^t))_{t \geq 0}$ is an Ornstein-Uhlenbeck process, i.e. a centered stationary Gaussian process with $E[U_tU_s] = \exp(-|t-s|/2)$.

- Known result (Slepian 1962,...):

$$P \left[ \sup_{t \in [0,T]} U_t \leq 0 \right] = \frac{1}{\pi} \arcsin(e^{-T/2}) \sim \frac{1}{\pi} e^{-T/2}, \quad T \to \infty.$$
Universal lower bound \(0 = t_0 < t_1 < \cdots < t_N\):

\[
P \left[ \sup_{n=0,\ldots,N} B_{t_n} \leq 0 \right] \geq \prod_{n=1}^{N} P[B(t_n) - B(t_{n-1}) \leq 0] = 2^{-N}.
\]

Continuous time:

\[
P \left[ \sup_{t \in [0,T]} B(e^{\beta t}) \leq 0 \right] \asymp e^{-\beta T/2}.
\]

In particular, the exponential rate of decay for continuous time and discrete time does not coincide in general.
Proposition (Aurzada/B. 2011)

Let $U$ be the Ornstein-Uhlenbeck process. Then

$$\lim_{N \to \infty} -\frac{1}{N} \log P \left[ \sup_{n=0, \ldots, N} U_{\beta n} \leq 0 \right] = \lambda_{\beta}.$$ 

Moreover, $\beta \mapsto \lambda_{\beta}$ is increasing and for all $\beta > \beta_0$, it holds that

$$C_1 e^{-\beta/2} \leq \log(2) - \lambda_{\beta} \leq C_2 e^{-\beta/2}.$$
Idea of proof

- Slepian’s inequality: $X$ centered Gaussian process such that $E[X_tX_s] \geq 0$. Then for $t_1 < \cdots < t_{N+M}$ and $x \in \mathbb{R}$

$$P \left[ \bigcap_{n=1}^{N+M} \{X_{t_n} \leq x\} \right] \geq P \left[ \bigcap_{n=1}^{N} \{X_{t_n} \leq x\} \right] P \left[ \bigcap_{n=N+1}^{N+M} \{X_{t_n} \leq x\} \right].$$

- For the stationary Ornstein-Uhlenbeck process, this implies for $t_n = \beta n$ that

$$N \mapsto \log P \left[ \sup_{n=0, \ldots, N} U_{\beta n} \leq 0 \right]$$

is subadditive.
Universality

- Let $Z_n = \sum_{k=1}^{n} e^{\beta k} X_k$ with $P[X_k = 1] = P[X_k = -1] = 1/2$.
- If $\beta > \log 2$:

  $$\sup_{n=1,\ldots,N} Z_n \leq 0 \iff X_1 = \cdots = X_N = -1.$$ 

- In particular,

  $$P \left[ \sup_{n=1,\ldots,N} Z_n \leq 0 \right] = 2^{-N} = e^{-\log(2)N}, \quad N \geq 1, \beta > \log 2.$$ 

- Gaussian case: $\lambda_\beta < \log 2$ for any $\beta$. 
Summary

- **Polynomial case**
  - Computation of the survival exponent in the Gaussian case.
  - Survival exponent is the same in the discrete and continuous time framework.
  - Universality of the survival exponent for a larger class of WRW (exponential moment condition).

- **Exponential case**
  - No universality.
  - Different survival exponents in continuous and discrete time framework.
  - Bounds on the rate of decay in the Gaussian case.
Thank you for your attention!