# Survival probabilities of weighted random walks

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- Problem: Given a centered process $(Z_t)_{t\geq 0}$, determine asymptotics of

$$p(T) := P\left[\sup_{t\in[0,T]} Z_t \leq 1\right], \quad T \to \infty.$$

- Typical: $p(T) \asymp T^{-\theta}$ or $p(T) = T^{-\theta+o(1)}$, $\theta$ is called survival exponent.
- $f \asymp g$ $(T \to \infty)$ means that $f(t)/g(t)$ is bounded away from zero and infinity for all $t$ large enough.
- Related problem:

$$P\left[\sup_{t\in[0,1]} Z_t \leq \epsilon\right], \quad \epsilon \downarrow 0.$$

- Brownian motion: $\theta = 1/2$ (reflection principle):

$$P\left[\sup_{t\in[0,T]} B_t \le 1\right] = P\left[|B_T| \le 1\right] \sim \sqrt{\frac{2}{\pi}}\, T^{-1/2}.$$

- Centered random walks $(S_n)_{n\ge 1}$ with finite variance: $\theta = 1/2$:

$$P\left[\sup_{n=1,\dots,N} S_n \le 1\right] \sim c\, N^{-1/2}$$

- $X_1, X_2, \ldots$ sequence of i.i.d. random variables with $E[X_1] = 0$ and $E\left[X_1^2\right] = 1$.
- $\sigma\colon [0,\infty) \to [0,\infty)$ some function with $\sigma(T) \to \infty$ as $T \to \infty$.
- Weighted random walk (WRW) $Z = (Z_n)_{n\geq 1}$ defined as

$$Z_n := \sum_{k=1}^{n} \sigma(k) X_k.$$

- Goal: determine survival exponent of $Z$ for a large class of functions $\sigma$.
- $E[Z_n Z_m] = t_n \wedge t_m$ where $t_k = \sigma(1)^2 + \cdots + \sigma(k)^2$. If $X_k \sim \mathcal{N}(0,1)$ i.i.d.:

$$(Z_n)_{n\geq 1} \stackrel{d}{=} (B_{t_n})_{n\geq 1}.$$

- Gaussian case: consider

$$P\left[\sup_{n=1,\ldots,N} B_{\kappa(n)} \leq 1\right], \quad N \to \infty.$$

## Theorem (Aurzada/B. 2011)

*Assume that* $\kappa(N) \asymp N^q$ *for some* $q > 0$ *and* $\kappa(N+1) - \kappa(N) \precsim N^\delta$ *for some* $\delta < q$. *Then*

$$P\left[\sup_{n=1,\dots,N} B_{\kappa(n)} \leq 1\right] = N^{-q/2+o(1)}.$$

- In particular,

$$P\left[\sup_{n=1,\dots,N} B_{\kappa(n)} \leq 1\right] = P\left[\sup_{t\in[0,\kappa(N)]} B_t \leq 1\right] N^{o(1)}.$$

- Extension possible to functions $\kappa(n) = \exp(n^\alpha)$ if $0 < \alpha < 1/4$.

# Sketch of proof

- Lower bound:
  $\{B_t \leq 1, \forall t \in [0, \kappa(N)]\} \subseteq \{B_{\kappa(n)} \leq 1, \forall n \leq N\}$
- Upper bound: One can find $\gamma \in (0, 1/2)$ and $\alpha > 0$ s.t.

$$P\left[\sup_{n=1,\ldots,N} B_{\kappa(n)} \leq 1\right] \leq P\left[\sup_{t\in[(\log N)^\alpha,\kappa(N)]} B_t - t^\gamma \leq 1\right] + o(N^{-q/2}).$$

  - For $c \in \mathbb{R}$ and $\gamma < 1/2$, it holds that (Uchiyama 1980)

$$P\left[\sup_{t\in[0,T]} B_t - ct^\gamma \leq 1\right] \asymp T^{-1/2}, \quad T \to \infty,$$

  - Slepian's inequality.

# Extension to WRW: Polynomial case

- $Z_n = \sum_{k=1}^n \sigma(k) X_k$, $\sigma(N) \asymp N^p$.
- Correpsonds to $\kappa(n) = \sum_{k=1}^n \sigma(k)^2 \asymp n^{2p+1}$ in the Gaussian case. Moreover, $\kappa(n+1) - \kappa(n) = \sigma(n+1)^2 \asymp n^{2p} \Rightarrow$ survival exponent is $p + 1/2$

**Theorem (Aurzada/B. 2011)**

*Let $(X_k)_{k \geq 1}$ be a sequence of i.i.d. centered random variables with $E\left[X_1^2\right] = 1$. Let $\sigma(N) \asymp N^p$ for some $p > 0$. If $E\left[|X_1|^\alpha\right] < \infty$ for some $\alpha > 4p + 2$, then*

$$P\left[\sup_{n=1,\dots,N} Z_n \leq 1\right] \gtrsim N^{-(p+1/2)}, \quad N \to \infty.$$

# Idea of proof

- Apply a Skorokhod embedding to the martingale $(Z_n)_{n\geq 1}$: there is an increasing sequence of stopping times $(\tau(n))_{n\geq 0}$ and a Brownian motion s.t. $(B_{\tau(n)})_n \overset{d}{=} (Z_n)_n$ and $(B_{t\wedge\tau(n)})_{t\geq 0}$ is uniformly integrable for all $n$.
- $$E\left[\tau(n)\right] = E\left[B_{\tau(n)}^2\right] = E\left[Z_n^2\right] = \sum_{k=1}^{n} \sigma(k)^2.$$
- $$P\left[\sup_{n=1,\dots,N} Z_n \leq 1\right] = P\left[\sup_{n=1,\dots,N} B_{\tau(n)} \leq 1\right]$$

# Extension to WRW: Polynomial case

## Theorem (Aurzada/B. 2011)

*Let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. centered random variables. Let $\sigma$ be increasing and $\sigma(N) \asymp N^p$ for some $p > 0$. If $E\left[e^{\alpha|X_1|}\right] < \infty$ for some $\alpha > 0$, then*

$$P\left[\sup_{n=1,\dots,N} Z_n \leq 1\right] \precsim N^{-(p+1/2)+o(1)}, \quad N \to \infty.$$

- The proof relies on a coupling of Komlós, Major and Tusnády (1976) that allows to reduce the problem to the Gaussian case.

$$P\left[\sup_{n=1,\dots,N}\left|\sum_{k=1}^{n} X_k - \sum_{k=1}^{n}\tilde{X}_k\right| \geq C\log N\right] \to 0.$$

# Gaussian framework with exponential weight function

- Consider for $\beta > 0$

$$P\left[\sup_{n=0,\dots,N} B(e^{\beta n}) \leq 0\right] = P\left[\sup_{n=0,\dots,N} U_{\beta n} \leq 0\right].$$

- Recall that $U = (e^{-t/2} B(e^t))_{t \geq 0}$ is an Ornstein-Uhlenbeck process, i.e. a centered stationary Gaussian process with $E\left[U_t U_s\right] = \exp(-\left|t-s\right|/2)$.
- Known result (Slepian 1962,...):

$$P\left[\sup_{t \in [0,T]} U_t \leq 0\right] = \frac{1}{\pi} \arcsin(e^{-T/2}) \sim \frac{1}{\pi} e^{-T/2}, \quad T \to \infty.$$

Universal lower bound ($0 = t_0 < t_1 < \cdots < t_N$):

$$P\left[\sup_{n=0,\dots,N} B_{t_n} \le 0\right] \ge \prod_{n=1}^{N} P[B(t_n) - B(t_{n-1}) \le 0] = 2^{-N}.$$

Continuous time:

$$P\left[\sup_{t\in[0,T]} B(e^{\beta t}) \le 0\right] \asymp e^{-\beta T/2}.$$

In particular, the exponential rate of decay for continuous time and discrete time does not coincide in general.

# The discrete Ornstein-Uhlenbeck process

## Proposition (Aurzada/B. 2011)

*Let $U$ be the Ornstein-Uhlenbeck process. Then*

$$\lim_{N\to\infty} -\frac{1}{N}\log P\left[\sup_{n=0,\ldots,N} U_{\beta n} \leq 0\right] = \lambda_\beta.$$

*Moreover, $\beta \mapsto \lambda_\beta$ is increasing and for all $\beta > \beta_0$, it holds that*

$$C_1 e^{-\beta/2} \leq \log(2) - \lambda_\beta \leq C_2 e^{-\beta/2}.$$

# Idea of proof

- Slepian's inequality: $X$ centered Gaussian process such that $E\left[X_t X_s\right] \geq 0$. Then for $t_1 < \cdots < t_{N+M}$ and $x \in \mathbb{R}$

$$P\left[\bigcap_{n=1}^{N+M} \{X_{t_n} \leq x\}\right] \geq P\left[\bigcap_{n=1}^{N} \{X_{t_n} \leq x\}\right] P\left[\bigcap_{n=N+1}^{N+M} \{X_{t_n} \leq x\}\right].$$

- For the stationary Ornstein-Uhlenbeck process, this implies for $t_n = \beta n$ that

$$N \mapsto \log P\left[\sup_{n=0,\ldots,N} U_{\beta n} \leq 0\right]$$

is subadditive.

# Universality

- Let $Z_n = \sum_{k=1}^{n} e^{\beta k} X_k$ with $P[X_k = 1] = P[X_k = -1] = 1/2$.
- If $\beta > \log 2$:

$$\sup_{n=1,\ldots,N} Z_n \leq 0 \Longleftrightarrow X_1 = \cdots = X_N = -1.$$

- In particular,

$$P\left[\sup_{n=1,\ldots,N} Z_n \leq 0\right] = 2^{-N} = e^{-\log(2)N}, \quad N \geq 1, \beta > \log 2.$$

- Gaussian case: $\lambda_\beta < \log 2$ for any $\beta$.

# Summary

- Polynomial case
  - Computation of the survival exponent in the Gaussian case.
  - Survival exponent is the same in the discrete and continuous time framework.
  - Universality of the survival exponent for a larger class of WRW (exponential moment condition).
- Exponential case
  - No universality.
  - Different survival exponents in continuous and discrete time framework.
  - Bounds on the rate of decay in the Gaussian case.

- Thank you for your attention!